

to appear in *Time-Frequency Signal Analysis and Processing*,  
B. Boashash, Ed., Englewood Cliffs (NJ): Prentice Hall, 2002

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## 4.7 TIME-FREQUENCY TRANSFER FUNCTION CALCULUS OF LINEAR TIME-VARYING SYSTEMS<sup>0</sup>

### 4.7.1 Linear Time-Varying Systems

Due to their generality, *linear time-varying* (LTV) systems (which can equivalently be viewed as linear operators [10]) have important advantages over linear time-invariant (LTI) systems. Applications of LTV systems include mobile communications (see Articles 9.5, 13.2, and 13.3), machine monitoring (see Article 15.2), and nonstationary statistical signal processing (see Articles 9.2, 9.4, 12.1, and 12.4). An LTV system  $\mathbf{H}$  maps the input signal  $x(t)$  to an output signal  $y(t)$  according to

$$y(t) = (\mathbf{H}x)(t) = \int_{-\infty}^{\infty} h(t, t') x(t') dt', \quad (4.7.1)$$

where  $h(t, t')$  is the kernel (impulse response) of  $\mathbf{H}$ . LTI systems and their dual, *linear frequency-invariant* (LFI) systems, are special cases of LTV systems. For an LTI system,  $y(t) = (\mathbf{H}x)(t) = \int_{-\infty}^{\infty} g(t-t') x(t') dt'$  and thus  $h(t, t') = g(t-t')$ . For an LFI system,  $y(t) = (\mathbf{H}x)(t) = w(t) x(t)$  and thus  $h(t, t') = w(t) \delta(t-t')$ .

For LTI and LFI systems, there exist physically intuitive and numerically efficient analysis and design methods that are based on the spectral transfer function  $G(f) = \int_{-\infty}^{\infty} g(\tau) e^{-j2\pi f\tau} d\tau$  and on the temporal transfer function  $w(t)$ , respectively. Unfortunately, a similar transfer function does not exist for LTV systems in general. However, in this article we will show that for the important class of *underspread* LTV systems, the *generalized Weyl symbol* constitutes an approximate time-frequency (TF) transfer function. We note that other TF symbols are discussed in Article 9.2.

### 4.7.2 The Generalized Weyl Symbol

The *generalized Weyl symbol* (GWS) of an LTV system  $\mathbf{H}$  is a family of linear TF representations defined as [5]

$$L_{\mathbf{H}}^{(\alpha)}(t, f) \triangleq \int_{-\infty}^{\infty} h^{(\alpha)}(t, \tau) e^{-j2\pi f\tau} d\tau$$

with

$$h^{(\alpha)}(t, \tau) \triangleq h\left(t + \left(\frac{1}{2} - \alpha\right)\tau, t - \left(\frac{1}{2} + \alpha\right)\tau\right), \quad (4.7.2)$$

where  $\alpha$  is a real-valued parameter. The GWS reduces to the ordinary *Weyl symbol* [3] [6] [8] [11] for  $\alpha = 0$ , to Zadeh's *time-varying transfer function* [12] for  $\alpha = 1/2$ ,

<sup>0</sup>Authors: **G. Matz** and **F. Hlawatsch**, Institute of Communications and Radio-Frequency Engineering, Vienna University of Technology, Gusshausstrasse 25/389, A-1040 Vienna, Austria (email: g.matz@ieee.org, fhlawats@pop.tuwien.ac.at, web: <http://www.nt.tuwien.ac.at/dspgroup/time.html>). Reviewers: S. Barbarossa and A. Papandreou-Suppappola. This work was supported by FWF grant P11904-TEC.

and to Bello's *frequency-dependent modulation function* [2] (also known as *Kohn-Nirenberg symbol* [3]) for  $\alpha = -1/2$ :

$$\begin{aligned} L_{\mathbf{H}}^{(0)}(t, f) &= \int_{-\infty}^{\infty} h\left(t + \frac{\tau}{2}, t - \frac{\tau}{2}\right) e^{-j2\pi f\tau} d\tau \\ L_{\mathbf{H}}^{(1/2)}(t, f) &= \int_{-\infty}^{\infty} h(t, t - \tau) e^{-j2\pi f\tau} d\tau \\ L_{\mathbf{H}}^{(-1/2)}(t, f) &= \int_{-\infty}^{\infty} h(t + \tau, t) e^{-j2\pi f\tau} d\tau. \end{aligned}$$

In what follows,  $\alpha$  will be considered fixed.

The GWS  $L_{\mathbf{H}}^{(\alpha)}(t, f)$  is a *linear* TF representation of the LTV system  $\mathbf{H}$ . It contains all information about  $\mathbf{H}$  since the kernel of  $\mathbf{H}$  can be recovered from the GWS:

$$h(t, t') = \int_{-\infty}^{\infty} L_{\mathbf{H}}^{(\alpha)}\left(\left(\frac{1}{2} + \alpha\right)t + \left(\frac{1}{2} - \alpha\right)t', f\right) e^{j2\pi f(t-t')} df.$$

Also, the input-output relation (4.7.1) can be reformulated in terms of the GWS. This reformulation becomes especially simple for  $\alpha = \pm 1/2$ :

$$y(t) = \int_{-\infty}^{\infty} L_{\mathbf{H}}^{(1/2)}(t, f) X(f) e^{j2\pi ft} df, \quad Y(f) = \int_{-\infty}^{\infty} L_{\mathbf{H}}^{(-1/2)}(t, f) x(t) e^{-j2\pi ft} dt.$$

For a rank-one system with impulse response  $h(t, t') = u(t) u^*(t')$ ,  $L_{\mathbf{H}}^{(\alpha)}(t, f)$  reduces to the generalized Wigner distribution [4] of the signal  $u(t)$ . Other interesting properties of the GWS can be found in [5] [7] [8] and for  $\alpha = 0$  in [3] [6] [11].

Next, we consider the GWS of some simple specific systems. The results obtained suggest that (under appropriate assumptions to be discussed later) the GWS can be interpreted as a "TF transfer function" that characterizes the "TF weighting" produced by the LTV system  $\mathbf{H}$ , i.e., the way in which a component of the input signal  $x(t)$  located about some TF point  $(t, f)$  is attenuated ( $|L_{\mathbf{H}}^{(\alpha)}(t, f)| < 1$ ), amplified ( $|L_{\mathbf{H}}^{(\alpha)}(t, f)| > 1$ ), or passed without attenuation or amplification ( $|L_{\mathbf{H}}^{(\alpha)}(t, f)| = 1$ ) by  $\mathbf{H}$ .

- The GWS of the identity operator  $\mathbf{I}$  with kernel  $h(t, t') = \delta(t - t')$  is given by  $L_{\mathbf{I}}^{(\alpha)}(t, f) \equiv 1$  (i.e., no attenuation/amplification anywhere in the TF plane).
- The GWS of the *TF shift operator*  $\mathfrak{S}_{\nu, \tau}^{(\alpha)}$  defined by<sup>1</sup>  $(\mathfrak{S}_{\nu, \tau}^{(\alpha)} x)(t) = x(t - \tau) e^{j2\pi \nu t} e^{j2\pi \nu \tau (\alpha - 1/2)}$  is a two-dimensional complex sinusoid<sup>2</sup>,  $L_{\mathfrak{S}_{\nu, \tau}^{(\alpha)}}^{(\alpha)}(t, f) =$

<sup>1</sup>The parameter  $\alpha$  in  $\mathfrak{S}_{\nu, \tau}^{(\alpha)}$  corresponds to the infinitely many ways of defining a joint TF shift by combining time shifts and frequency shifts. In particular,  $\alpha = 1/2$  corresponds to first shifting in time and then shifting in frequency, whereas  $\alpha = -1/2$  corresponds to first shifting in frequency and then in time.

<sup>2</sup>Note that the GWS parameter  $\alpha$  is chosen equal to the parameter  $\alpha$  in  $\mathfrak{S}_{\nu, \tau}^{(\alpha)}$ .

$e^{j2\pi(\nu t - \tau f)}$ , and thus  $|L_{\mathfrak{H}}^{(\alpha)}(t, f)| = 1$  (i.e., no attenuation/amplification anywhere in the TF plane).

- The GWS of an LTI system with kernel  $h(t, t') = g(t - t')$  reduces to the ordinary transfer function  $G(f)$  for all  $t$ , i.e.,  $L_{\mathbf{H}}^{(\alpha)}(t, f) \equiv G(f)$ .
- The GWS of an LFI system with kernel  $h(t, t') = w(t) \delta(t - t')$  reduces to the temporal transfer function  $w(t)$  for all  $f$ , i.e.,  $L_{\mathbf{H}}^{(\alpha)}(t, f) \equiv w(t)$ .

The last two examples show the GWS's consistency with the conventional transfer functions.

In what follows, we shall investigate the conditions under which the GWS can be interpreted as a TF transfer function. The key element in this investigation is an analysis of the *TF shifts* produced by  $\mathbf{H}$ . For this analysis, we need another linear TF representation of  $\mathbf{H}$ , to be discussed next.

### 4.7.3 The Generalized Spreading Function

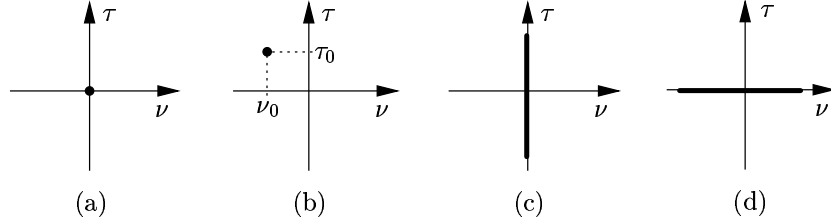
Besides a TF-dependent weighting, LTV systems can also introduce TF shifts of various input signal components. Here, the TF shift operator  $\mathfrak{H}_{\nu, \tau}^{(\alpha)}$  mentioned above—with  $\nu$  denoting frequency (Doppler) shift and  $\tau$  denoting time shift/delay—is an elementary example. A joint description of the TF shifts introduced by a linear system  $\mathbf{H}$  is given by the *generalized spreading function* (GSF) [5]

$$S_{\mathbf{H}}^{(\alpha)}(\nu, \tau) \triangleq \int_{-\infty}^{\infty} h^{(\alpha)}(t, \tau) e^{-j2\pi\nu t} dt,$$

with  $h^{(\alpha)}(t, \tau)$  as in (4.7.2). Like the GWS, the GSF  $S_{\mathbf{H}}^{(\alpha)}(\nu, \tau)$  is a *linear* TF representation of the LTV system  $\mathbf{H}$ , and it contains all information about  $\mathbf{H}$  since the kernel of  $\mathbf{H}$  can be recovered from the GWS. The input-output relation (4.7.1) can be reformulated in terms of the GSF according to

$$y(t) = (\mathbf{H}x)(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{\mathbf{H}}^{(\alpha)}(\nu, \tau) (\mathfrak{H}_{\nu, \tau}^{(\alpha)}x)(t) d\nu d\tau.$$

This represents the output signal  $y(t) = (\mathbf{H}x)(t)$  as a weighted superposition of TF shifted versions  $(\mathfrak{H}_{\nu, \tau}^{(\alpha)}x)(t) = x(t - \tau) e^{j2\pi\nu t} e^{j2\pi\nu\tau(\alpha-1/2)}$  of the input signal  $x(t)$ . The  $(\nu, \tau)$ -dependent weights in this superposition are given by the GSF, thus establishing the GSF's interpretation as a “TF shift distribution” of  $\mathbf{H}$ . The extension of  $S_{\mathbf{H}}^{(\alpha)}(\nu, \tau)$  about the origin of the  $(\nu, \tau)$  plane indicates the amount of TF shifts caused by  $\mathbf{H}$ . In particular, a large extension of  $S_{\mathbf{H}}^{(\alpha)}(\nu, \tau)$  in the  $\nu$  direction indicates large frequency/Doppler shifts (equivalently, fast time variation) and a large extension of  $S_{\mathbf{H}}^{(\alpha)}(\nu, \tau)$  in the  $\tau$  direction indicates large time shifts (delays).



**Figure 4.1:** Schematic representation of the GSF magnitude of some (classes of) linear systems: (a) identity operator, (b) TF shift operator, (c) LTI system, (d) LFI system.

GSFs with different  $\alpha$  values differ merely by a phase factor, i.e.,

$$S_{\mathbf{H}}^{(\alpha_2)}(\nu, \tau) = S_{\mathbf{H}}^{(\alpha_1)}(\nu, \tau) e^{j2\pi(\alpha_1 - \alpha_2)\nu\tau}.$$

Therefore, the GSF magnitude is independent of  $\alpha$ ,  $|S_{\mathbf{H}}^{(\alpha_1)}(\nu, \tau)| = |S_{\mathbf{H}}^{(\alpha_2)}(\nu, \tau)|$ , and thus we may simply write  $|S_{\mathbf{H}}(\nu, \tau)|$ . The GSF is related to the GWS by a two-dimensional Fourier transform,

$$S_{\mathbf{H}}^{(\alpha)}(\nu, \tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L_{\mathbf{H}}^{(\alpha)}(t, f) e^{-j2\pi(\nu t - \tau f)} dt df. \quad (4.7.3)$$

Further properties of the GSF are described in [5] [7] [8].

Again, it is instructive to consider a few examples (see Fig. 4.1):

- The GSF of the identity operator  $\mathbf{I}$  is given by  $S_{\mathbf{I}}^{(\alpha)}(\nu, \tau) = \delta(\nu)\delta(\tau)$ , which is zero for  $(\nu, \tau) \neq (0, 0)$  (i.e., neither frequency shifts nor time shifts).
- The GSF of a TF shift operator  $\mathfrak{S}_{\nu_0, \tau_0}^{(\alpha)}$  is obtained as  $S_{\mathfrak{S}_{\nu_0, \tau_0}^{(\alpha)}}^{(\alpha)}(\nu, \tau) = \delta(\nu - \nu_0)\delta(\tau - \tau_0)$ , which is zero for  $(\nu, \tau) \neq (\nu_0, \tau_0)$  (i.e., no frequency and time shifts other than by  $\nu_0$  and  $\tau_0$ , respectively).
- The GSF of an LTI system  $\mathbf{H}$  with kernel  $h(t, t') = g(t - t')$  is given by  $S_{\mathbf{H}}^{(\alpha)}(\nu, \tau) = \delta(\nu)g(\tau)$  (i.e., only time shifts whose distribution is characterized by the impulse response  $g(\tau)$ ).
- The GSF of an LFI system  $\mathbf{H}$  with  $h(t, t') = w(t)\delta(t - t')$  is given by  $S_{\mathbf{H}}^{(\alpha)}(\nu, \tau) = W(\nu)\delta(\tau)$ , where  $W(\nu)$  is the Fourier transform of  $w(t)$  (i.e., only frequency shifts whose distribution is characterized by  $W(\nu)$ ).

#### 4.7.4 Underspread LTV Systems

Using the GSF, we now define the class of *underspread* LTV systems [7] [8] for which, as we will see in Section 4.7.5, the GWS acts as a “TF transfer function.” Conceptually, an LTV system is underspread if its GSF is well concentrated about

the origin of the  $(\nu, \tau)$  plane, which indicates that the system introduces only small TF shifts, i.e., the system's time variations are slow and/or its memory is short. In contrast, systems introducing large TF shifts are termed *overspread* [7] [8].

There are two alternative mathematical characterizations of the GSF extension and, in turn, of underspread systems. The first characterization [7] requires that the support of the GSF  $S_{\mathbf{H}}^{(\alpha)}(\nu, \tau)$  is confined to a compact region  $\mathcal{G}_{\mathbf{H}}$  about the origin of the  $(\nu, \tau)$  plane, i.e.,  $|S_{\mathbf{H}}(\nu, \tau)| = 0$  for  $(\nu, \tau) \notin \mathcal{G}_{\mathbf{H}}$ . Let  $\nu_{\mathbf{H}} \triangleq \max_{(\nu, \tau) \in \mathcal{G}_{\mathbf{H}}} |\nu|$  and  $\tau_{\mathbf{H}} \triangleq \max_{(\nu, \tau) \in \mathcal{G}_{\mathbf{H}}} |\tau|$  denote the maximum frequency shift and time shift, respectively, introduced by the system  $\mathbf{H}$ . We define the *Doppler-delay spread* of  $\mathbf{H}$  as  $\sigma_{\mathbf{H}} \triangleq 4\nu_{\mathbf{H}}\tau_{\mathbf{H}}$ , which is the area of the rectangle  $[-\nu_{\mathbf{H}}, \nu_{\mathbf{H}}] \times [-\tau_{\mathbf{H}}, \tau_{\mathbf{H}}]$  enclosing  $\mathcal{G}_{\mathbf{H}}$ . Underspread LTV systems are then defined by the condition  $\sigma_{\mathbf{H}} \ll 1$ .

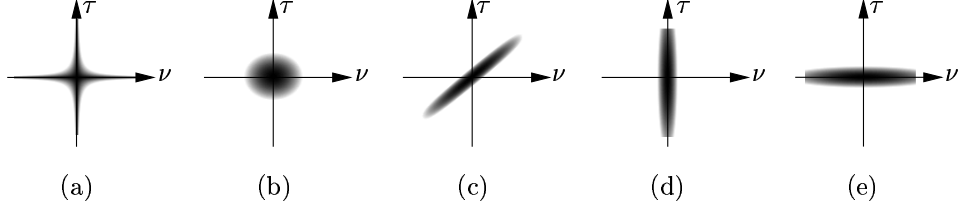
Unfortunately, the GSF of practical LTV systems rarely has compact support. An alternative, much more flexible characterization of the GSF extension and of underspread systems that does not require a compact support assumption is based on the normalized *weighted GSF integrals*<sup>3</sup> [8]

$$m_{\mathbf{H}}^{(\phi)} \triangleq \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(\nu, \tau) |S_{\mathbf{H}}(\nu, \tau)| d\nu d\tau}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |S_{\mathbf{H}}(\nu, \tau)| d\nu d\tau}.$$

Here,  $\phi(\nu, \tau)$  is a nonnegative weighting function that satisfies  $\phi(\nu, \tau) \geq \phi(0, 0) = 0$  and penalizes GSF contributions lying away from the origin. We also define the *GSF moments*  $m_{\mathbf{H}}^{(k,l)} \triangleq m_{\mathbf{H}}^{(\phi_{k,l})}$  as special cases of  $m_{\mathbf{H}}^{(\phi)}$  using the weighting functions  $\phi_{k,l}(\nu, \tau) = |\nu|^k |\tau|^l$  with  $k, l \in \mathbb{N}_0$ . Thus, without the assumption of compact GSF support, a system  $\mathbf{H}$  can now be considered to be underspread if suitable GSF integrals/moments are “small.” While this is not a clear-cut definition of underspread systems, it has the advantage of being more flexible than the previous definition that was based on the area of the (assumedly) compact support of the GSF. It can be shown that if the GSF of  $\mathbf{H}$  does have compact support with maximum frequency shift  $\nu_{\mathbf{H}}$  and maximum time shift  $\tau_{\mathbf{H}}$ , then  $m_{\mathbf{H}}^{(k,l)} \leq \nu_{\mathbf{H}}^k \tau_{\mathbf{H}}^l$  and, in particular,  $m_{\mathbf{H}}^{(k,k)} \leq (\sigma_{\mathbf{H}}/4)^k$ . Thus, LTV systems that are underspread in the compact-support sense can be considered as a special case of the extended underspread framework based on weighted GSF integrals and moments.

Examples of various types of underspread systems are illustrated in Fig. 4.2. It should be noted that the concept of underspread systems is not equivalent to that of slowly time-varying (quasi-LTI) systems. A quasi-LTI system (i.e., small  $m_{\mathbf{H}}^{(0,l)}$ ) may be overspread if its memory is very long (i.e., very large  $m_{\mathbf{H}}^{(k,0)}$ ), and a system with faster time-variations (i.e., larger  $m_{\mathbf{H}}^{(0,l)}$ ) may be underspread if its memory is short enough (i.e., very small  $m_{\mathbf{H}}^{(k,0)}$ ). Finally, note that according to (4.7.3), the GWS of an underspread LTV system is a *smooth* function.

<sup>3</sup>Other definitions of weighted GSF integrals and moments can be found in [8].



**Figure 4.2:** GSF magnitude of (a) an underspread system with small  $m_{\mathbf{H}}^{(1,1)}$ , (b) an underspread system with small  $m_{\mathbf{H}}^{(1,0)}$   $m_{\mathbf{H}}^{(0,1)}$ , (c) a “chirpy” underspread system [8], (d) a quasi-LTI system (slowly time-varying; small  $m_{\mathbf{H}}^{(0,1)}$ ), (e) a quasi-LFI system (short memory; small  $m_{\mathbf{H}}^{(1,0)}$ ).

### 4.7.5 Time-Frequency Transfer Function Calculus

For *underspread* LTV systems as defined in the previous section, the GWS acts as an approximate “TF transfer function” that generalizes the spectral (temporal) transfer function of LTI (LFI) systems. Indeed, if specific weighted GSF integrals  $m_{\mathbf{H}}^{(\phi)}$  and/or moments  $m_{\mathbf{H}}^{(k,l)}$  are small, one can show the validity of several transfer function approximations [7] [8], some of which are discussed in the following. Applications of these approximations include time-varying spectral analysis, linear TF filter design, and detection/estimation of nonstationary random processes (for references see Section 4.7.6).

**Adjoint system.** For an LTI system  $\mathbf{H}$  with transfer function  $G(f)$ , the transfer function of the *adjoint*  $\mathbf{H}^+$  is  $G^*(f)$ . A similar correspondence holds for LFI systems. For general LTV systems, the GWS of the adjoint  $\mathbf{H}^+$  (with kernel  $h^+(t,t') = h^*(t',t)$  [10]) is not equal to the conjugate of the GWS of  $\mathbf{H}$  unless  $\alpha = 0$ . However, for an underspread LTV system  $\mathbf{H}$  this is approximately true:

$$L_{\mathbf{H}^+}^{(\alpha)}(t, f) \approx [L_{\mathbf{H}}^{(\alpha)}(t, f)]^*. \quad (4.7.4)$$

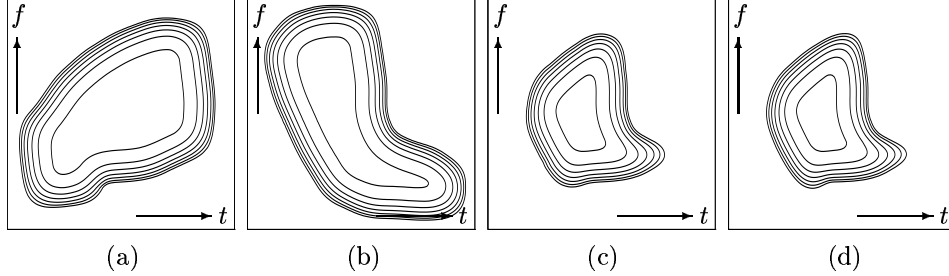
Indeed, it can be shown [8] that the associated approximation error is upper bounded as

$$|L_{\mathbf{H}^+}^{(\alpha)}(t, f) - [L_{\mathbf{H}}^{(\alpha)}(t, f)]^*| \leq 4\pi |\alpha| \|S_{\mathbf{H}}\|_1 m_{\mathbf{H}}^{(1,1)}, \quad (4.7.5)$$

with  $\|S_{\mathbf{H}}\|_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |S_{\mathbf{H}}(\nu, \tau)| d\nu d\tau$ . Thus, for an underspread system where  $m_{\mathbf{H}}^{(1,1)}$  is small, the approximation (4.7.4) will be quite good. It follows from (4.7.4) that the GWS of a self-adjoint, underspread LTV system is approximately real-valued:

$$\mathbf{H}^+ = \mathbf{H} \quad \Longrightarrow \quad [L_{\mathbf{H}}^{(\alpha)}(t, f)]^* \approx L_{\mathbf{H}}^{(\alpha)}(t, f), \quad \Im\{L_{\mathbf{H}}^{(\alpha)}(t, f)\} \approx 0.$$

In addition, if the underspread LTV system  $\mathbf{H}$  is positive (semi-)definite [10], then  $\Re\{L_{\mathbf{H}}^{(\alpha)}(t, f)\} \approx L_{\mathbf{H}}^{(\alpha)}(t, f)$  is approximately nonnegative.



**Figure 4.3:** Transfer function approximation of the Weyl symbol (GWS with  $\alpha = 0$ ) for a composition of underspread systems: (a) Weyl symbol of  $\mathbf{H}_1$ , (b) Weyl symbol of  $\mathbf{H}_2$ , (c) Weyl symbol of  $\mathbf{H}_2\mathbf{H}_1$ , and (d) product of the individual Weyl symbols  $L_{\mathbf{H}_1}^{(0)}(t, f)$  and  $L_{\mathbf{H}_2}^{(0)}(t, f)$ . Note the similarity of (c) and (d). In this and subsequent simulations, time duration is 128 samples and normalized frequency ranges from  $-1/4$  to  $1/4$ .

**Composition of systems.** The transfer function of a composition (series connection) of two LTI systems with transfer functions  $G_1(f)$  and  $G_2(f)$  is given by the product  $G_1(f)G_2(f)$ . An analogous result holds for the composition of two LFI systems. This composition property of transfer functions is the cornerstone of many signal processing techniques. Unfortunately, a similar composition property no longer holds true for the GWS in the case of general LTV systems. However, the GWS of the composition  $\mathbf{H}_2\mathbf{H}_1$  of two jointly underspread [8] LTV systems  $\mathbf{H}_1$  and  $\mathbf{H}_2$  is approximately equal to the product of their GWSs, i.e.,

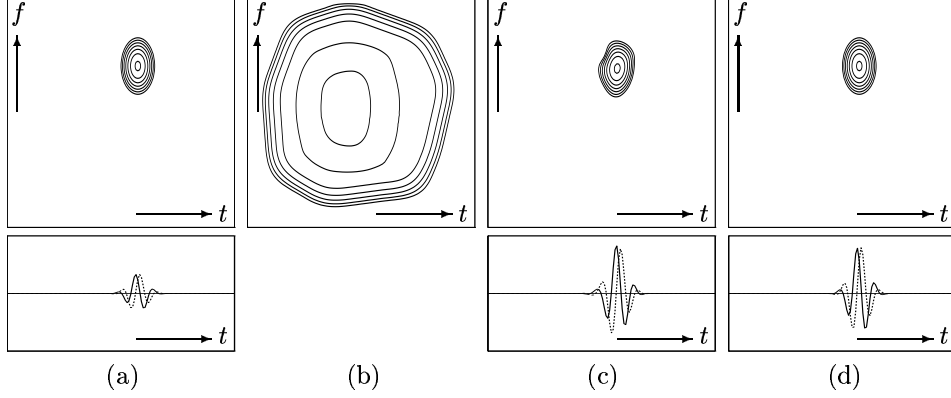
$$L_{\mathbf{H}_2\mathbf{H}_1}^{(\alpha)}(t, f) \approx L_{\mathbf{H}_1}^{(\alpha)}(t, f) L_{\mathbf{H}_2}^{(\alpha)}(t, f). \quad (4.7.6)$$

An upper bound on the associated approximation error (similar to (4.7.5)) can again be provided [8]. Combining (4.7.4) with (4.7.6), we furthermore obtain

$$L_{\mathbf{H}\mathbf{H}^+}^{(\alpha)}(t, f) \approx L_{\mathbf{H}+\mathbf{H}}^{(\alpha)}(t, f) \approx |L_{\mathbf{H}}^{(\alpha)}(t, f)|^2.$$

Fig. 4.3 shows an example illustrating the approximation (4.7.6) for  $\alpha = 0$ . In this example, the maximum normalized error  $\max_{t,f} |L_{\mathbf{H}_2\mathbf{H}_1}^{(0)}(t, f) - L_{\mathbf{H}_1}^{(0)}(t, f) L_{\mathbf{H}_2}^{(0)}(t, f)| / \max_{t,f} |L_{\mathbf{H}_2\mathbf{H}_1}^{(0)}(t, f)|$  is 0.045, which means that the approximation is quite good.

**Approximate eigenfunctions and eigenvalues.** The response of an LTI system with transfer function  $G(f)$  to a complex sinusoid  $e^{j2\pi f_0 t}$  is  $G(f_0) e^{j2\pi f_0 t}$ , and the response of an LFI system with temporal transfer function  $w(t)$  to a Dirac impulse  $\delta(t - t_0)$  is given by  $w(t_0) \delta(t - t_0)$ . Hence, complex sinusoids and Dirac impulses are the eigenfunctions of LTI and LFI systems, respectively, with the eigenvalues given by corresponding values of the transfer function,  $G(f_0)$  and  $w(t_0)$ . In contrast, the eigenfunctions of general LTV systems are not localized or structured in any sense. However, for underspread LTV systems, well TF localized functions are approximate eigenfunctions and the GWS  $L_{\mathbf{H}}^{(\alpha)}(t, f)$  constitutes an approximate



**Figure 4.4:** Eigenfunction/eigenvalue approximation of the Weyl symbol (GWS with  $\alpha = 0$ ) of an underspread LTV system: (a) Wigner distribution [9] (top) and real and imaginary parts (bottom) of input signal  $s_{t_0, f_0}(t)$ , (b) Weyl symbol of  $\mathbf{H}$ , (c) output signal  $(\mathbf{H}s_{t_0, f_0})(t)$ , and (d) input signal  $s_{t_0, f_0}(t)$  multiplied by  $L_{\mathbf{H}}(t_0, f_0)$ . Note the similarity of (c) and (d).

eigenvalue distribution over the TF plane (for related results see [1] and Article 13.3). Indeed, consider the following family of signals,

$$s_{t_0, f_0}(t) = s(t - t_0) e^{j2\pi f_0 t},$$

where  $s(t)$  is a signal well TF localized about  $t = 0$  and  $f = 0$ . Evidently,  $s_{t_0, f_0}(t)$  is well TF localized about  $t = t_0$  and  $f = f_0$ . For an underspread LTV system  $\mathbf{H}$ , one can show [8]

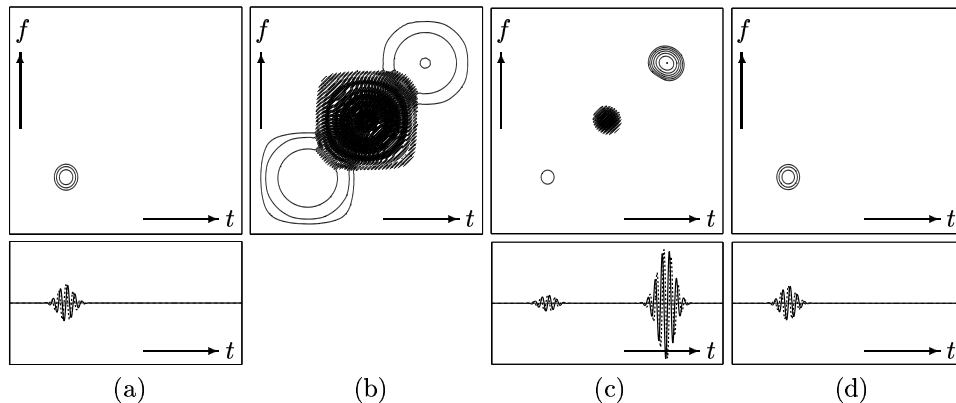
$$(\mathbf{H}s_{t_0, f_0})(t) \approx L_{\mathbf{H}}^{(\alpha)}(t_0, f_0) s_{t_0, f_0}(t), \quad (4.7.7)$$

i.e., the signals  $s_{t_0, f_0}(t)$  are “approximate eigenfunctions” of  $\mathbf{H}$ , with the associated “approximate eigenvalues” given by the GWS values  $L_{\mathbf{H}}^{(\alpha)}(t_0, f_0)$ . Eq. (4.7.7) shows that a signal well TF localized about some TF point is passed by  $\mathbf{H}$  nearly undistorted; it is merely weighted by the GWS value at that point, thus corroborating the GWS’s interpretation as a *TF transfer function*. An example illustrating the approximation (4.7.7) for the case  $\alpha = 0$  is shown in Fig. 4.4. In this example, the normalized error  $\|\mathbf{H}s_{t_0, f_0} - L_{\mathbf{H}}^{(0)}(t_0, f_0) s_{t_0, f_0}\|^2 / \|\mathbf{H}s_{t_0, f_0}\|^2$  (with  $\|x\|^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt$ ) is 0.06.

**Approximate uniqueness of the GWS.** For an underspread LTV system  $\mathbf{H}$ , it can furthermore be shown [8] that the GWS is only weakly dependent on the parameter  $\alpha$ . That is,

$$L_{\mathbf{H}}^{(\alpha_1)}(t, f) \approx L_{\mathbf{H}}^{(\alpha_2)}(t, f)$$

for moderate values of  $|\alpha_1 - \alpha_2|$ . This means that the TF transfer function provided by the GWS is approximately *unique*. In particular, the Weyl symbol  $L_{\mathbf{H}}^{(0)}(t, f)$ , Zadeh’s time-varying transfer function  $L_{\mathbf{H}}^{(1/2)}(t, f)$ , and Bello’s frequency-dependent



**Figure 4.5:** Violation of the eigenfunction/eigenvalue approximation of the Weyl symbol in the case of an overspread LTV system: (a) Wigner distribution (top) and real and imaginary parts (bottom) of input signal  $s_{t_0, f_0}(t)$ , (b) Weyl symbol of  $\mathbf{H}$ , (c) output signal  $(\mathbf{H}s_{t_0, f_0})(t)$ , and (d) input signal  $s_{t_0, f_0}(t)$  multiplied by  $L_{\mathbf{H}}(t_0, f_0)$ . This figure should be compared with Fig. 4.4; note that now the signals in (c) and (d) are very different. The overspread character of  $\mathbf{H}$  is indicated by the rapid oscillation of the Weyl symbol in (b) and by the fact—evident upon comparison of (a) and (c)—that the signal  $s_{t_0, f_0}(t)$  is partly TF shifted by the system  $\mathbf{H}$ .

modulation function  $L_{\mathbf{H}}^{(-1/2)}(t, f)$  will be approximately equivalent for an underspread LTV system  $\mathbf{H}$ .

**Discussion.** As mentioned before, the above approximate relations (more can be found in [7] [8]) extend analogous (exact) relations satisfied by the conventional transfer function of LTI/LFI systems. In this sense, the GWS is an approximate TF transfer function of underspread LTV systems. As a mathematical underpinning of these approximations, explicit upper bounds on the associated approximation errors have been developed [7] [8]. These bounds are formulated in terms of the GSF parameters  $m_{\mathbf{H}}^{(\phi)}$ ,  $m_{\mathbf{H}}^{(k, l)}$  or  $\sigma_{\mathbf{H}}$  defined in Section 4.7.4. If specific such parameters are small (indicating that  $\mathbf{H}$  is underspread), the upper bounds on specific approximation errors are small and thus the respective approximation will be good.

On the other hand, we caution that the above approximations and, thus, the GWS's interpretation as a TF transfer function are *not* valid for overspread LTV systems. This is illustrated in Fig. 4.5.

#### 4.7.6 Summary and Conclusions

While general linear, time-varying systems are fairly difficult to work with, we have shown that for the practically important class of *underspread* systems a very simple and intuitively appealing time-frequency transfer function calculus can be developed. Indeed, the generalized Weyl symbol can be used as an approximate *time-frequency transfer function* in a similar way as the conventional transfer function of time-invariant systems. Applications of this time-frequency transfer function

calculus to nonstationary signal analysis and processing are considered in Articles 9.4, 11.1, and 12.4. Finally, an extension to *random* systems/channels is considered in Article 9.5.

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