

Time-Frequency Transfer Function Calculus (Symbolic Calculus) of Linear Time-Varying Systems (Linear Operators) Based on a Generalized Underspread Theory*

Gerald Matz and Franz Hlawatsch

*Department of Communications and Radio-Frequency Engineering, Vienna University of Technology
Gusshausstr. 25/389, A-1040 Wien, Austria
phone: +43 1 58801 3515, fax: +43 1 5870583, email: gmatz@aurora.nt.tuwien.ac.at
WWW: <http://www.nt.tuwien.ac.at/dspgroup/time.html>*

Dedicated to Professor Wolfgang Mecklenbräuer on the occasion of his sixtieth birthday.

We introduce a generalized concept of *underspread* linear time-varying systems (linear operators) which contains a previous definition as a special case. We show that an existing approximate transfer function calculus (symbolic calculus) can be extended to this wider and practically more relevant class of underspread operators. As a mathematical underpinning of this calculus, we establish explicit bounds on various error quantities associated with it. The transfer function calculus provides a theoretical basis for various methods recently proposed for nonstationary signal processing, and it has important implications in the theory of time-varying power spectra.

I. INTRODUCTION

Quantum physics and time-frequency signal theory are strongly interrelated fields. In a formal sense, the phase space of physics (physical quantities momentum p and position q) plays a similar role in quantum mechanics as the time-frequency plane in signal theory (variables time t and frequency f). This formal analogy has led to numerous interactions between the two fields. Important examples are the Wigner distribution^{1–9}, the uncertainty principle^{10–15}, the connections between coherent states and the short-time Fourier transform or Gabor expansion^{9,12,15–24}, and the relation of the Zak (or Weil-Brezin) transform with the polyphase decomposition^{15,20,25–30}. However, notwithstanding their similarity on a formal mathematical level, the interpretation and scope of time-frequency signal analysis and semiclassical quantum mechanics are somewhat different:

- In the semiclassical approach to quantum mechanics, one desires a “mock phase space”³¹ that is similar to the phase space of classical mechanics. In contrast, in signal analysis one desires joint time-frequency descriptions (of signals, systems, random processes, ...) in order to combine pure time-domain and pure frequency-domain descriptions. More specifically, the phase-space descriptions of quantum observables (corresponding to self-adjoint operators on Hilbert spaces) are intended to preserve as much as possible the structural properties and conceptual power of the phase-space description of classical observables (functions in phase space). An example is the computation of expectations in phase space using quasi-probability densities. On the other hand, in signal analysis there is nothing comparable to classical observables (therefore, taking the classical limit $\hbar \rightarrow 0$ does not make sense in signal analysis). Rather, one is interested in extending existing powerful concepts like the transfer function/frequency response of time-invariant linear systems and the power spectral density of stationary random processes to the time-varying and nonstationary cases, respectively. These time-frequency representations should retain as much as possible the conceptual power and intuitive interpretation of the transfer function (power spectral density).
- The difficulties encountered when working with a quantum-mechanical mock phase space arise mainly from the fundamental mathematical difference between functions (classical observables) and operators (quantum-mechanical observables). In contrast, the main complications in the time-frequency description of time-varying systems (nonstationary random processes) are due to their fundamental difference from time-invariant systems (stationary random processes) for which a pure frequency-domain description is sufficient and satisfactory.

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- Another difference between the viewpoints taken in quantum mechanics and in signal analysis concerns the non-commutativity of linear operators. In quantum mechanics, this non-commutativity is a fundamental fact that is part of our mathematical description of the physical world, whereas in signal analysis it can be considered simply as an inconvenience that prevents powerful, simple, and intuitive concepts and methods valid in the time-invariant/stationary case from being extended to the more general time-varying/nonstationary case. In a signal analysis context, therefore, it is reasonable and practically important to ask whether there exists a subclass of operators (time-varying systems) that is approximately commuting (cf. Subsections III C and IV A).

A. Summary of Results and Outline of Paper

This paper considers a time-frequency description of linear operators or, equivalently, linear time-varying systems that is related to semiclassical quantum mechanics^{2,15,32–36} and to the mathematical theory of pseudo-differential operators^{37–41}. This time-frequency description of operators is an attempt to generalize the simple and efficient frequency-domain description of linear time-invariant systems—namely, the transfer function or frequency response—to the time-varying case. The result is a transfer function calculus (known as *symbolic calculus* in a quantum-mechanical context) that extends a previously introduced transfer function calculus^{42–44} to a considerably wider and practically more relevant class of linear operators. This extended transfer function calculus provides a theoretical basis for several methods that have recently been proposed for nonstationary signal processing (specifically, to design and implement time-varying filters for signal enhancement, estimation, and detection^{45–53}). Furthermore, it has important implications in the theory of time-varying power spectra for nonstationary random processes^{42,53–55}.

The transfer function calculus presented is based on the generalized Weyl symbol which is discussed in Subsection IC. This discussion is preceded by a brief review of time-varying, time-invariant, and frequency-invariant linear operators in Subsection IB.

As a basis for our extended transfer function calculus, Section II develops a generalized concept of “underspread” operators that uses weighted integrals and moments of the operators’ spreading function and does not require the spreading function to have compact support. Subsequently, in Section III we employ these integrals/moments to formulate explicit bounds on the errors incurred by the transfer function approximation. These bounds provide a rigorous mathematical underpinning for the extended, approximate transfer function calculus proposed. We note that several of these bounds and their proofs are analogous to similar bounds and proofs presented previously^{42–44}, even though our formulation using weighted integrals and moments of the spreading function is new. Finally, Section IV shows that underspread operators are approximately commuting and normal.

We note that symbolic calculi in a similar spirit have been obtained in the context of semiclassical quantum mechanics^{15,33,39,56} and pseudo-differential operators^{15,37,38,40}, with particular similarity to classical asymptotics of quantum mechanics^{10,15,39,56–58} (like the WBK method¹⁰ or Voros’ h -expansions^{39,56}). The difference is that these theories define specific symbol classes—whose mathematical sophistication has hitherto prevented their application to signal processing problems—directly in phase space whereas in our approach we formulate growth conditions in the (dual) spreading function domain (cf. Subsection IIB). Also, whereas in quantization and pseudo-differential operator theory one studies the operators corresponding to a given symbol, we consider a given linear operator and investigate how close its generalized Weyl symbol comes to the intuitive engineering notion of a time-frequency transfer function.

B. Time-Varying, Time-Invariant, and Frequency-Invariant Linear Operators

Linear¹ time-varying systems/operators are useful models in a variety of engineering applications as diverse as speech production and mobile communications. Except for the limiting cases of time-invariant and frequency-invariant systems (see below), we will subsequently restrict our attention to Hilbert-Schmidt operators⁵⁹, i.e., operators \mathbf{H} with finite Hilbert-Schmidt norm

$$\|\mathbf{H}\|_2 \triangleq \sqrt{\sum_k \|\mathbf{H}u_k\|_2^2} < \infty,$$

¹In what follows, we tacitly assume that all systems/operators are linear.

where $\{u_k(t)\}$ is an arbitrary orthonormal basis of $L_2(\mathbb{R})$. Although in quantum mechanics many observables correspond to unbounded operators, our restriction to Hilbert-Schmidt operators (which are always bounded) is consistent with most engineering applications. The input-output relation for a Hilbert-Schmidt operator \mathbf{H} can be written as (integrals are from $-\infty$ to ∞)

$$(\mathbf{H}x)(t) = \int_{t'} h(t, t') x(t') dt', \quad (1)$$

where $x(t)$ and $(\mathbf{H}x)(t)$ are the system's input and output signal, respectively, and $h(t, t') \in L_2(\mathbb{R}^2)$ is the kernel (impulse response) of \mathbf{H} . We note that $\|\mathbf{H}\|_2^2 = \int_t \int_{t'} |h(t, t')|^2 dt dt'$.

Unfortunately, time-varying systems/operators are much more difficult to analyze and characterize than time-invariant systems, i.e., operators that commute with time shifts and have a convolution-type kernel of the form $h(t, t') = g(t - t')$. For time-invariant systems the *transfer function* (frequency response)

$$G(f) = \int_{\tau} g(\tau) e^{-i2\pi f\tau} d\tau$$

is an extremely simple and efficient system description. This is due to the following properties:

- The complex sinusoids $\{e^{i2\pi ft}\}$ (parametrized by frequency f) are the generalized⁶⁰ eigenfunctions of *any* time-invariant operator, with $G(f)$ the associated generalized eigenvalues, i.e. $(\mathbf{H}x)(t) = G(f) e^{i2\pi ft}$ for $x(t) = e^{i2\pi ft}$. This is also the reason why all linear time-invariant systems commute with each other.
- As a consequence of the previous property, the Fourier transform of $(\mathbf{H}x)(t)$ equals the Fourier transform of $x(t)$ multiplied by $G(f)$. Hence, in the frequency (Fourier) domain, the input-output relation simplifies to the multiplication of two functions.
- The transfer function of the series connection (product) of two linear time-invariant systems with transfer functions $G_1(f)$ and $G_2(f)$ equals $G_1(f) G_2(f)$.
- The adjoint⁵⁹ \mathbf{H}^+ of a time-invariant system has impulse response $g^*(-\tau)$, and hence its transfer function is simply the complex conjugate of $G(f)$.
- For input signals $x(t) \in L_2(\mathbb{R})$, the maximum system gain is equal to the supremum of $|G(f)|$.

The dual of linear time-invariant systems is given by linear “frequency-invariant” systems (i.e., operators commuting with frequency shifts), which correspond to time-domain multiplications and have an impulse response of the form $h(t, t') = m(t) \delta(t - t')$. Here, the factor $m(t)$ plays the role of a “temporal transfer function” with properties similar to those of the spectral transfer function $G(f)$ of linear time-invariant systems. We note that the quantum-mechanical counterparts of time-invariant and frequency-invariant systems are observables that depend solely on momentum and solely on position, respectively.

Compared to time-invariant and frequency-invariant systems, general linear operators are much more difficult and less intuitive to work with. In particular, as opposed to time-invariant (frequency-invariant) operators which are always normal (i.e., which satisfy $\mathbf{H}^+\mathbf{H} = \mathbf{H}\mathbf{H}^+$) and which always commute, general operators are potentially non-normal and typically non-commuting. For normal operators, a generalization of the transfer function $G(f)$ or $m(t)$ is provided by the operator's eigenvalues⁵⁹ (in the non-normal case one has to recur to singular values^{59,61}). For example, the adjoint operator and the operator norm (maximum system gain) correspond to the complex conjugate and supremum, respectively, of the eigenvalues. Unfortunately, for practical engineering applications this concept has certain drawbacks and limitations:

- The eigenfunctions of a general linear operator are not known *a priori*, and their computation is expensive. The lack of a simple structure of the eigenfunctions prevents the application of efficient algorithms like the fast Fourier transform that is useful in the time-invariant case.
- The eigenfunctions and eigenvalues are not related to the frequency or time variable.
- The eigenfunctions of two non-commuting operators are different, and the eigenvalues of the operators' product (series connection of systems) is not equal to the product of the operators' eigenvalues.

C. The Generalized Weyl Symbol

Since the eigenvalue-based transfer function concept does not yield the desired physical relevance and computational efficiency, an alternative “transfer function-like” characterization of linear operators is needed. A “time-varying transfer function” of time-varying systems was introduced by Zadeh⁶² and later complemented by Bello’s *frequency-dependent modulation function*⁶³ which is equivalent to the *Kohn-Nirenberg symbol*^{15,37} introduced in the theory of pseudo-differential operators. These two definitions correspond to what is known in the physical literature as standard and anti-standard correspondence rules^{31,34–36}. In a signal processing context, the *Weyl symbol*^{4,15,32,64,65}, originally introduced in quantum mechanics⁶⁶, was then recognized as a further definition of a “transfer function”⁶⁷.

Finally, Kozek^{42,44,68} placed all three definitions within a unifying framework by defining the *generalized Weyl symbol* as

$$L_{\mathbf{H}}^{(\alpha)}(t, f) \triangleq \int_{\tau} h^{(\alpha)}(t, \tau) e^{-i2\pi f\tau} d\tau, \quad (2)$$

with the following coordinate-transformed version of the impulse response $h(t, t')$ in (1),

$$h^{(\alpha)}(t, \tau) \triangleq h\left(t + \left(\frac{1}{2} - \alpha\right)\tau, t - \left(\frac{1}{2} + \alpha\right)\tau\right). \quad (3)$$

Here, α is a parameter for which an interpretation will be provided in Subsection II A. Although any real-valued number may be taken for α , the case $|\alpha| \leq 1/2$ is of particular importance since this condition guarantees that the generalized Weyl symbol satisfies finite support properties⁴². We note that the generalized Weyl symbol is a special case of a more general correspondence rule between classical and quantum observables^{31,34,35,69}. It is a function of the physically relevant variables time t and frequency f (in the quantum mechanical context, these correspond to position q and momentum p). For $\alpha = 0, 1/2$, and $-1/2$, it reduces to respectively the Weyl symbol, Zadeh’s time-varying transfer function, and Bello’s frequency-dependent modulation function. For rank-one operators, i.e., operators \mathbf{H} of the type $h(t, t') = s(t) s^*(t')$, it equals the generalized Wigner distribution^{70,71} of $s(t)$. For $\alpha = 0$ the generalized Wigner distribution simplifies to the Wigner distribution^{1–6,8,9} whereas for $\alpha = 1/2$ the Rihaczek distribution⁷² is obtained.

The generalized Weyl symbol satisfies many desirable mathematical properties^{42,68}. In particular, it preserves inner products and Hilbert-Schmidt norms of operators, its integral over the entire time-frequency plane yields the operator’s trace, and it is “covariant” to time-frequency shifts (Galilean invariance) and time-frequency scalings of operators. Furthermore, for time-invariant and frequency-invariant operators it simplifies to the spectral transfer function $G(f)$ and the temporal transfer function $m(t)$, respectively (this is the signal analysis reformulation of the fact that any valid quantum-mechanical correspondence rule yields the position and momentum operators as counterparts of the classical observables position and momentum, respectively).

The Weyl symbol (obtained with $\alpha = 0$) possesses an inherent symmetry which leads to two further important properties:

- The Weyl symbol of the adjoint operator \mathbf{H}^+ equals the complex conjugate of the Weyl symbol of \mathbf{H} , i.e., $L_{\mathbf{H}^+}^{(0)}(t, f) = L_{\mathbf{H}}^{(0)*}(t, f)$. Hence, the Weyl symbol of a self-adjoint (Hermitian) operator, i.e., an operator satisfying $\mathbf{H}^+ = \mathbf{H}$, is real-valued^{15,31,32,36}.
- The Weyl symbol is covariant to operator transformations by metaplectic operators $\mathbf{U} \in \mathcal{M}$, i.e., unitary operators that belong to the metaplectic representation \mathcal{M} of the symplectic group $\text{Sp}(2, \mathbb{R})$ consisting of measure-preserving, linear time-frequency coordinate transforms $\begin{pmatrix} t \\ f \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t \\ f \end{pmatrix}$ (with $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = 1$) such as rotations, shearings, and scalings.^{4,15,40,41,64,73–75}

$$L_{\mathbf{U}\mathbf{H}\mathbf{U}^+}^{(0)}(t, f) = L_{\mathbf{H}}^{(0)}(at + bf, ct + df). \quad (4)$$

Unfortunately, in spite of its attractive mathematical properties the generalized Weyl symbol cannot, in general, be used as a transfer function in a manner similar to the time-invariant or frequency-invariant case. For example, the generalized Weyl symbol of an operator product does not generally equal the product of the individual symbols, and the supremum of the magnitude of the generalized Weyl symbol does not generally reflect the operator norm. Indeed, in a signal analysis context it has often been conjectured that the concept of a time-varying transfer function requires the system to be *slowly time-varying*^{48,63,76–80}. However, it seems that until recently no quantitative formulation of this conjecture has been available.

In his pioneering work^{42–44}, Kozek defined a class of “underspread” linear operators which even includes operators with fast time variations. He showed that this class allows an approximate transfer function calculus (symbolic calculus) based on the generalized Weyl symbol. Underspread operators were defined by the condition that the operator’s spreading function (see Section II) has compact support contained within a rectangle of area $\ll 1$. Unfortunately, in practice this is typically not satisfied exactly but only effectively.

II. EXTENDED CONCEPT OF UNDERSPREAD SYSTEMS

Our extension of the generalized Weyl symbol based transfer function calculus (symbolic calculus) is based on a generalized concept of underspread operators. This concept is related to the time-frequency shifts introduced by linear operators and their characterization by means of the spreading function.

A. The Spreading Function

Contrary to time-invariant systems (which cause only time shifts) and frequency-invariant systems (which cause only frequency shifts), general linear operators shift the input signal with respect to both time and frequency. This behavior is reflected by the following reformulation of the input-output relation (1) as a weighted superposition of time-frequency-shifted versions of the input signal^{42–44,63,64,68,81}

$$(\mathbf{H}x)(t) = \int_{\tau} \int_{\nu} S_{\mathbf{H}}^{(\alpha)}(\tau, \nu) x_{\tau, \nu}^{(\alpha)}(t) d\tau d\nu. \quad (5)$$

Here, $x_{\tau, \nu}^{(\alpha)}(t) = x(t - \tau) e^{i2\pi\nu t} e^{i2\pi(\alpha-1/2)\tau\nu}$ is the signal $x(t)$ shifted by τ in time and by ν in frequency, with $\alpha \in \mathbb{R}$ expressing the freedom in ordering the time and frequency shifts^{32,42,68}. This generalized time-frequency shift is equivalent to the Schrödinger representation of the 3-D Heisenberg group^{15,42,75,82} with parameters $(-\tau, \nu, \alpha\tau\nu)$. Relation (5) is furthermore equivalent to Weyl’s original idea for establishing a correspondence rule between classical and quantum observables by Fourier-transforming the classical observable and then replacing the position and momentum variables in the exponent by the position and momentum operators^{15,31,32,34,36,83}. The “weighting function” in (5), $S_{\mathbf{H}}^{(\alpha)}(\tau, \nu)$, is the (*generalized*) *spreading function* of \mathbf{H} which is defined as^{15,42,63,68,81}

$$S_{\mathbf{H}}^{(\alpha)}(\tau, \nu) \triangleq \int_t h^{(\alpha)}(t, \tau) e^{-i2\pi\nu t} dt,$$

with $h^{(\alpha)}(t, \tau)$ as in (3). For $\alpha = \pm 1/2$, the spreading function has first been introduced under the names of delay-Doppler spread function and Doppler-delay spread function by Bello⁶³ who considered communication channels with multipath propagation and moving scatterers. Here, the received signal $(\mathbf{H}x)(t)$ consists of several delayed and Doppler-shifted versions of the original signal with the spreading function describing the “reflectivity” of the individual scatterers.

The spreading function is the 2-D (symplectic) Fourier transform of the generalized Weyl symbol in (2),

$$S_{\mathbf{H}}^{(\alpha)}(\tau, \nu) = \int_t \int_f L_{\mathbf{H}}^{(\alpha)}(t, f) e^{-i2\pi(\nu t - \tau f)} dt df. \quad (6)$$

Spreading functions with distinct α differ only by a phase factor⁴²,

$$S_{\mathbf{H}}^{(\alpha_2)}(\tau, \nu) = S_{\mathbf{H}}^{(\alpha_1)}(\tau, \nu) e^{i2\pi(\alpha_1 - \alpha_2)\tau\nu}, \quad (7)$$

so that $|S_{\mathbf{H}}^{(\alpha_2)}(\tau, \nu)| = |S_{\mathbf{H}}^{(\alpha_1)}(\tau, \nu)|$. Hence, we will write $|S_{\mathbf{H}}(\tau, \nu)|$ instead of $|S_{\mathbf{H}}^{(\alpha)}(\tau, \nu)|$.

Like the generalized Weyl symbol, the spreading function preserves inner products and Hilbert-Schmidt norms of operators. Its value at the origin equals the operator’s trace, and its magnitude is covariant to metaplectic transformations $\mathbf{U} \in \mathcal{M}$ (cf. Subsection I C),

$$|S_{\mathbf{U}\mathbf{H}\mathbf{U}^+}(\tau, \nu)| = |S_{\mathbf{H}}(a\tau + b\nu, c\tau + d\nu)|. \quad (8)$$

For a time-invariant system, we have $S_{\mathbf{H}}^{(\alpha)}(\tau, \nu) = g(\tau) \delta(\nu)$ which is perfectly concentrated along the τ axis. Similarly, for a frequency-invariant system, $S_{\mathbf{H}}^{(\alpha)}(\tau, \nu) = M(\nu) \delta(\tau)$ (with $M(\nu)$ the Fourier transform of $m(t)$) which is perfectly concentrated along the ν axis (cf. Fig. 1). This is in accordance with the fact that time-invariant systems cause only time shifts and frequency-invariant systems cause only frequency shifts.

B. Measures for the Spread of the Spreading Function

The spread, or extension, of the spreading function about the origin² of the (τ, ν) plane provides a global characterization of the time-frequency shifts introduced by the operator. The extension in the τ direction characterizes the operator's "length of memory" whereas the extension in the ν direction determines the fastness of the operator's time-variations or fluctuations. The mathematical formulation of the concept of *underspread operators* (cf. Subsection II C) requires quantitative measures of the extension of the spreading function about the origin. For operators with compactly supported spreading function, such a measure is provided by the area of the smallest rectangle containing the support of the spreading function⁴²⁻⁴⁴. In practice, however, the compact support condition is often not met. To circumvent this problem, we here propose to use the following *weighted integrals* of the spreading function $S_{\mathbf{H}}^{(\alpha)}(\tau, \nu)$ which are normalized by the L_1 or L_2 norm³ of $S_{\mathbf{H}}^{(\alpha)}(\tau, \nu)$:

$$\begin{aligned} m_{\mathbf{H}}^{(\phi)} &\triangleq \frac{1}{\|S_{\mathbf{H}}\|_1} \int_{\tau} \int_{\nu} \phi(\tau, \nu) |S_{\mathbf{H}}(\tau, \nu)| d\tau d\nu, \\ M_{\mathbf{H}}^{(\phi)} &\triangleq \frac{1}{\|\mathbf{H}\|_2} \left[\int_{\tau} \int_{\nu} \phi^2(\tau, \nu) |S_{\mathbf{H}}(\tau, \nu)|^2 d\tau d\nu \right]^{1/2}. \end{aligned} \quad (9)$$

Here, $\phi(\tau, \nu)$ is a nonnegative weighting function which satisfies $\phi(\tau, \nu) \geq \phi(0, 0) = 0$ and thus penalizes contributions of the spreading function lying away from the origin. We note that $m_{\mathbf{H}}^{(\phi)}$ and $M_{\mathbf{H}}^{(\phi)}$ are nonnegative and do not depend on the parameter α . Since the spreading function of the adjoint operator \mathbf{H}^+ is given by $S_{\mathbf{H}^+}^{(\alpha)}(\tau, \nu) = S_{\mathbf{H}}^{(-\alpha)*}(-\tau, -\nu)$, we have $m_{\mathbf{H}^+}^{(\phi)} = m_{\mathbf{H}}^{(\phi)}$ and $M_{\mathbf{H}^+}^{(\phi)} = M_{\mathbf{H}}^{(\phi)}$ if the weighting function is even-symmetric, i.e., $\phi(-\tau, -\nu) = \phi(\tau, \nu)$. Fig. 2 shows some specific weighting functions that will be used in Sections III and IV.

We also introduce the (*absolute*) *moments* $m_{\mathbf{H}}^{(k,l)}$ and $M_{\mathbf{H}}^{(k,l)}$ of the spreading function as a special case of the above weighted integrals using weighting functions $\phi(\tau, \nu) = |\tau|^k |\nu|^l$ with $k, l \in \mathbb{N}_0$:

$$\begin{aligned} m_{\mathbf{H}}^{(k,l)} &\triangleq \frac{1}{\|S_{\mathbf{H}}\|_1} \int_{\tau} \int_{\nu} |\tau|^k |\nu|^l |S_{\mathbf{H}}(\tau, \nu)| d\tau d\nu, \\ M_{\mathbf{H}}^{(k,l)} &\triangleq \frac{1}{\|\mathbf{H}\|_2} \left[\int_{\tau} \int_{\nu} \tau^{2k} \nu^{2l} |S_{\mathbf{H}}(\tau, \nu)|^2 d\tau d\nu \right]^{\frac{1}{2}}. \end{aligned}$$

We have $m_{\mathbf{H}^+}^{(k,l)} = m_{\mathbf{H}}^{(k,l)}$ and $M_{\mathbf{H}^+}^{(k,l)} = M_{\mathbf{H}}^{(k,l)}$ and furthermore

$$0 \leq m_{\mathbf{H}}^{(k,l)} \leq \sqrt{m_{\mathbf{H}}^{(2k,0)} m_{\mathbf{H}}^{(0,2l)}}, \quad 0 \leq M_{\mathbf{H}}^{(k,l)} \leq \sqrt{M_{\mathbf{H}}^{(2k,0)} M_{\mathbf{H}}^{(0,2l)}}. \quad (10)$$

Moments with $k = 0$ or $l = 0$ penalize mainly contributions of the spreading function that are located away from the τ axis or away from the ν axis, respectively. Thus, operators with spreading function concentrated along the τ axis (i.e., quasi-time-invariant operators, see Fig. 1(d)) have small $m_{\mathbf{H}}^{(0,l)}$ and small $M_{\mathbf{H}}^{(0,l)}$ (see Fig. 2(b)), whereas operators with spreading function concentrated along the ν axis (i.e., quasi-frequency-invariant operators, see Fig. 1(e)) have small $m_{\mathbf{H}}^{(k,0)}$ and small $M_{\mathbf{H}}^{(k,0)}$ (see Fig. 2(a)). Moments with $k = l$ penalize mainly contributions of the spreading function that are located away from the τ and ν axes, i.e., that lie in oblique directions in the (τ, ν) plane. This is due to the fact that the corresponding weighting function is constant along the hyperbolae $|\tau\nu| = c$ (cf. Fig. 2(c)). In particular, a superposition (i.e., parallel connection) of a (quasi-) time-invariant operator and a (quasi-) frequency-invariant operator has a spreading function concentrated along the τ and ν axes and will thus have small $m_{\mathbf{H}}^{(k,k)}$ and $M_{\mathbf{H}}^{(k,k)}$.

²If the spreading function is concentrated about some point $(\tau_0, \nu_0) \neq (0, 0)$, this corresponds to an offset time-frequency shift which can be split off from the operator, resulting in an operator whose spreading function is concentrated about the origin.

³These norms are defined as $\|S_{\mathbf{H}}\|_1 = \int_{\tau} \int_{\nu} |S_{\mathbf{H}}(\tau, \nu)| d\tau d\nu$ (which is assumed to be finite) and $\|S_{\mathbf{H}}\|_2^2 = \int_{\tau} \int_{\nu} |S_{\mathbf{H}}(\tau, \nu)|^2 d\tau d\nu$. We note that $\|S_{\mathbf{H}}\|_2 = \|\mathbf{H}\|_2$.

C. Underspread Operators

Conceptually, a linear operator is called *underspread* if its spreading function is concentrated in a small region about the origin of the (τ, ν) plane, which indicates that the operator introduces only small time-frequency shifts τ, ν or, in other words, that the operator's memory is short and/or its time-variations are slow.

The concept of underspread operators was first used for *random* linear time-varying systems in the context of fading multipath channels^{63,78,81,84,85} where physical considerations (path loss, limited velocities) naturally suggested limitations of delay (τ) and Doppler shifts (ν). Adopting the terminology of the random case, Kozek⁴²⁻⁴⁴ introduced underspread deterministic linear operators by requiring that their spreading function be *exactly* zero outside a small rectangular support region about the origin. In practice, however, this condition is often not satisfied exactly but only *effectively*. This raises the question of how to choose the effective support region and how the modeling error resulting from a specific choice of effective support region affects the validity of the results obtained using the compact support model.

These problems and limitations can be circumvented by using the weighted integrals $m_{\mathbf{H}}^{(\phi)}$ and $M_{\mathbf{H}}^{(\phi)}$ and, specifically, the moments $m_{\mathbf{H}}^{(k,l)}$ and $M_{\mathbf{H}}^{(k,l)}$ introduced in Subsection II B. Hence, without being forced to assume that the spreading function has finite support, we can consider an operator \mathbf{H} to be *underspread* if suitable integrals/moments of the spreading function are “small.” We note that this is not a clear-cut definition of underspread operators. Indeed, we will see in Section III that bounds on various error quantities associated with the transfer function approximation are formulated using different integrals/moments of the spreading function. Hence, having these approximation errors small calls for different integrals/moments being small. This is somewhat similar to the many different quantities that can be used to measure the effective bandwidth of a signal. Our concept of underspread operators is thus more complicated and diffuse, but also more flexible and practically relevant, than Kozek's definition which required the spreading function to have (small) compact support. It is precisely this flexibility which, in Section III, will allow us to establish bounds for the transfer function calculus (symbolic calculus) approximation without unnecessarily restrictive assumptions.

If the spreading function is exactly zero outside a finite region \mathcal{S} around the origin, the weighted integrals are bounded as $m_{\mathbf{H}}^{(\phi)} \leq \phi_{\max}$ and $M_{\mathbf{H}}^{(\phi)} \leq \phi_{\max}$ with $\phi_{\max} = \max_{(\tau, \nu) \in \mathcal{S}} \{\phi(\tau, \nu)\}$. In particular, for rectangular regions $\mathcal{S} = [-\tau_{\max}, \tau_{\max}] \times [-\nu_{\max}, \nu_{\max}]$, one has $m_{\mathbf{H}}^{(k,l)} \leq \tau_{\max}^k \nu_{\max}^l$ and $M_{\mathbf{H}}^{(k,l)} \leq \tau_{\max}^k \nu_{\max}^l$. Thus, operators that are underspread in the sense of Kozek⁴²⁻⁴⁴, i.e., that have compact support contained within a rectangle of area $4\tau_{\max}\nu_{\max} \ll 1$, are a special case of our framework.

It should be noted that the concept of underspread operators is not equivalent to that of slowly time-varying (quasi-time-invariant) operators which requires $|S_{\mathbf{H}}(\tau, \nu)|$ to be narrow with respect to ν (i.e., small moments $m_{\mathbf{H}}^{(0,l)}$ and $M_{\mathbf{H}}^{(0,l)}$, see Fig. 1(d) and Fig. 2(b)). In particular, a slowly time-varying operator may be overspread (i.e., not underspread) if its memory is very long, and an operator with fast time-variations may be underspread if its memory is short enough.

Due to the 2-D Fourier transform relation (6) between the spreading function and the generalized Weyl symbol, the generalized Weyl symbol of an underspread operator is a *smooth* (i.e. a 2-D lowpass-type) function. This can also be seen from the following relations between the partial derivatives of the generalized Weyl symbol and the moments of the spreading function:

$$\left| \frac{\partial^{k+l} L_{\mathbf{H}}^{(\alpha)}(t, f)}{\partial t^l \partial f^k} \right| \leq (2\pi)^{k+l} \|S_{\mathbf{H}}\|_1 m_{\mathbf{H}}^{(k,l)}, \quad \left\| \frac{\partial^{k+l} L_{\mathbf{H}}^{(\alpha)}}{\partial t^l \partial f^k} \right\|_2 = (2\pi)^{k+l} \|\mathbf{H}\|_2 M_{\mathbf{H}}^{(k,l)}. \quad (11)$$

In particular, $\left| \frac{\partial L_{\mathbf{H}}^{(\alpha)}(t, f)}{\partial t} \right|$ will be small—implying smoothness of the generalized Weyl symbol in the time direction—if the moments $m_{\mathbf{H}}^{(0,1)}$ and $M_{\mathbf{H}}^{(0,1)}$ are small. Similarly, the generalized Weyl symbol will be smooth in the frequency direction (i.e. $\left| \frac{\partial L_{\mathbf{H}}^{(\alpha)}(t, f)}{\partial f} \right|$ will be small) if $m_{\mathbf{H}}^{(1,0)}$ and $M_{\mathbf{H}}^{(1,0)}$ are small. We note that the first relation (bound) in (11) bears some resemblance to the definition of specific symbol classes^{15,37,38,40,41,56} of pseudo-differential operators, with the difference that our bound is uniform over the entire time-frequency plane whereas in pseudo-differential operator theory there is an additional term controlling the growth of the partial derivatives of the symbol.

III. TRANSFER FUNCTION CALCULUS (SYMBOLIC CALCULUS)

In this section, we show that for linear operators that are underspread in the extended sense of Subsection II C, the generalized Weyl symbol $L_{\mathbf{H}}^{(\alpha)}(t, f)$ is an approximate “time-frequency transfer function” that generalizes

the spectral/temporal transfer function of time-invariant/frequency-invariant linear systems. As a mathematical underpinning of this approximate time-frequency transfer function calculus (symbolic calculus), we establish explicit upper bounds on the errors associated with it. These bounds are formulated in terms of the integrals and/or moments of the spreading function introduced in Subsection IIB; they do not require the spreading function to have finite support. Hence, our subsequent results will show that a transfer function calculus (symbolic calculus) is valid for a significantly wider class of systems than that considered previously^{42–44}. This calculus leads to simple and numerically efficient time-frequency methods for specific nonstationary signal processing applications^{45–53} and it yields important results on time-varying power spectra of nonstationary random processes^{42,54,55}.

A. Approximate Uniqueness of the Generalized Weyl Symbol

The dependence of the generalized Weyl symbol on the parameter α is a difference from the time-invariant and frequency-invariant cases where the transfer function of a given operator is uniquely defined. In a signal processing context, this dependence may be considered an inconvenience since in practice it is often not clear which α value should be selected. In particular, the case $\alpha = 0$ features an important symplectic covariance (cf. (4)) whereas the cases $\alpha = \pm 1/2$ lend themselves to simpler discrete-time implementations. Fortunately, the following bounds on the generalized Weyl symbol's α -dependence (which extend existing bounds⁴²) show that in the *underspread* case the choice of α is not critical.

Theorem III.1. *For any operator \mathbf{H} , the difference $\Delta_1^{(\alpha_1, \alpha_2)}(t, f) \triangleq L_{\mathbf{H}}^{(\alpha_1)}(t, f) - L_{\mathbf{H}}^{(\alpha_2)}(t, f)$ between two generalized Weyl symbols with parameters α_1 and α_2 is bounded as⁴*

$$\frac{|\Delta_1^{(\alpha_1, \alpha_2)}(t, f)|}{\|\mathbf{S}_{\mathbf{H}}\|_1} \leq 2\pi|\alpha_1 - \alpha_2| m_{\mathbf{H}}^{(1,1)}, \quad \frac{\|\Delta_1^{(\alpha_1, \alpha_2)}\|_2}{\|\mathbf{H}\|_2} \leq 2\pi|\alpha_1 - \alpha_2| M_{\mathbf{H}}^{(1,1)}. \quad (12)$$

Proof. With (6) and (7), the 2-D Fourier transform of $\Delta_1^{(\alpha_1, \alpha_2)}(t, f)$ is given by

$$\hat{\Delta}_1^{(\alpha_1, \alpha_2)}(\tau, \nu) = \int_t \int_f \Delta_1^{(\alpha_1, \alpha_2)}(t, f) e^{-i2\pi(\nu t - \tau f)} dt df = S_{\mathbf{H}}^{(\alpha_1)}(\tau, \nu) \left[1 - e^{i2\pi(\alpha_1 - \alpha_2)\tau\nu} \right].$$

The first bound (L_{∞} bound) in (12) is then shown as

$$\begin{aligned} |\Delta_1^{(\alpha_1, \alpha_2)}(t, f)| &= \left| \int_{\tau} \int_{\nu} \hat{\Delta}_1^{(\alpha_1, \alpha_2)}(\tau, \nu) e^{i2\pi(\nu t - \tau f)} d\tau d\nu \right| \leq \int_{\tau} \int_{\nu} |\hat{\Delta}_1^{(\alpha_1, \alpha_2)}(\tau, \nu)| d\tau d\nu \\ &= \int_{\tau} \int_{\nu} |S_{\mathbf{H}}(\tau, \nu)| \left| 1 - e^{i2\pi(\alpha_1 - \alpha_2)\tau\nu} \right| d\tau d\nu = 2 \int_{\tau} \int_{\nu} |S_{\mathbf{H}}(\tau, \nu)| |\sin(\pi(\alpha_1 - \alpha_2)\tau\nu)| d\tau d\nu \\ &\leq 2\pi|\alpha_1 - \alpha_2| \int_{\tau} \int_{\nu} |S_{\mathbf{H}}(\tau, \nu)| |\tau| |\nu| d\tau d\nu = 2\pi|\alpha_1 - \alpha_2| \|\mathbf{S}_{\mathbf{H}}\|_1 m_{\mathbf{H}}^{(1,1)}, \end{aligned}$$

where we used $|\sin x| \leq |x|$. The second bound (L_2 bound) is shown by noting that

$$\|\Delta_1^{(\alpha_1, \alpha_2)}\|_2^2 = \|\hat{\Delta}_1^{(\alpha_1, \alpha_2)}\|_2^2 = \int_{\tau} \int_{\nu} |S_{\mathbf{H}}(\tau, \nu)|^2 \left| 1 - e^{i2\pi(\alpha_1 - \alpha_2)\tau\nu} \right|^2 d\tau d\nu$$

and proceeding similarly as above, using $\sin^2 x \leq x^2$. □

Note that the bounds in this theorem depend only on the difference $\alpha_1 - \alpha_2$ and not on α_1, α_2 individually. For $|\alpha| \leq 1/2$, one obtains the coarser but α -independent bounds $2\pi m_{\mathbf{H}}^{(1,1)}$ and $2\pi M_{\mathbf{H}}^{(1,1)}$. For an *underspread*

⁴Here, $\|\Delta_1^{(\alpha_1, \alpha_2)}\|_2$ is the L_2 norm of $\Delta_1^{(\alpha_1, \alpha_2)}(t, f)$ defined as $\|\Delta_1^{(\alpha_1, \alpha_2)}\|_2^2 = \int_t \int_f |\Delta_1^{(\alpha_1, \alpha_2)}(t, f)|^2 dt df$. We shall consider bounds on the *normalized* error magnitude and L_2 norm. The magnitude is normalized by $\|\mathbf{S}_{\mathbf{H}}\|_1$ since $|L_{\mathbf{H}}^{(\alpha)}(t, f)| \leq \|\mathbf{S}_{\mathbf{H}}\|_1$, and the L_2 norm is normalized by $\|\mathbf{H}\|_2$ since $\|L_{\mathbf{H}}^{(\alpha)}\|_2 = \|\mathbf{H}\|_2$. Furthermore, in this and subsequent bounds the integrals and moments involved are assumed to exist.

operator whose moments $m_{\mathbf{H}}^{(1,1)}$ and $M_{\mathbf{H}}^{(1,1)}$ are small, the bounds in (12) show that the generalized Weyl symbol is approximately independent of α , i.e.,

$$L_{\mathbf{H}}^{(\alpha_1)}(t, f) \approx L_{\mathbf{H}}^{(\alpha_2)}(t, f),$$

as long as $|\alpha_1 - \alpha_2|$ is not too large. Hence, the time-frequency transfer function of an underspread operator is approximately unique. We note that small $m_{\mathbf{H}}^{(1,1)}$ and $M_{\mathbf{H}}^{(1,1)}$ requires the spreading function to be concentrated along the τ and/or ν axes (i.e., not oriented in oblique directions). Important examples are quasi-time-invariant and quasi-frequency-invariant systems and parallel connections (i.e. sums) thereof. Since small $m_{\mathbf{H}}^{(1,1)}$ and $M_{\mathbf{H}}^{(1,1)}$ does not require the operator's spreading function to be exactly supported within a rectangular region about the origin, the bounds in (12) are more general than previously derived bounds⁴². For example, $m_{\mathbf{H}}^{(1,1)}$ and $M_{\mathbf{H}}^{(1,1)}$ will be small for an operator whose spreading function is concentrated within the region $|\tau\nu| \leq c$ with small c (cf. Fig. 1(f)). We also note that results in a similar spirit have been derived by Kohn and Nirenberg³⁷ for the special case $\alpha_1 = 1/2$ and $\alpha_2 = -1/2$ and by Folland¹⁵ for the special case $\alpha_1 = 0$ and $\alpha_2 = 1/2$. Finally, we recall from (10) that $m_{\mathbf{H}}^{(1,1)} \leq \sqrt{m_{\mathbf{H}}^{(2,0)}m_{\mathbf{H}}^{(0,2)}}$ and $M_{\mathbf{H}}^{(1,1)} \leq \sqrt{M_{\mathbf{H}}^{(2,0)}M_{\mathbf{H}}^{(0,2)}}$.

B. Adjoint Operator

The kernel of the adjoint \mathbf{H}^+ of a linear operator \mathbf{H} with kernel $h(t, t')$ is given by $h^*(t', t)$. In the special case of a time-invariant or frequency-invariant operator, the transfer function of the adjoint operator is obtained by complex conjugation of the original transfer function. In contrast, the generalized Weyl symbol of the adjoint of a general linear operator is easily shown to be $L_{\mathbf{H}^+}^{(\alpha)}(t, f) = L_{\mathbf{H}}^{(-\alpha)*}(t, f)$, which does not equal $L_{\mathbf{H}}^{(\alpha)*}(t, f)$ unless $\alpha = 0$. However, the following bounds are obtained from Theorem III.1.

Corollary III.2. *For any operator \mathbf{H} , the difference $\Delta_2^{(\alpha)}(t, f) \triangleq L_{\mathbf{H}^+}^{(\alpha)}(t, f) - L_{\mathbf{H}}^{(\alpha)*}(t, f)$ is bounded as*

$$\frac{|\Delta_2^{(\alpha)}(t, f)|}{\|S_{\mathbf{H}}\|_1} \leq 4\pi|\alpha| m_{\mathbf{H}}^{(1,1)}, \quad \frac{\|\Delta_2^{(\alpha)}\|_2}{\|\mathbf{H}\|_2} \leq 4\pi|\alpha| M_{\mathbf{H}}^{(1,1)}. \quad (13)$$

Proof. With $L_{\mathbf{H}^+}^{(\alpha)}(t, f) = L_{\mathbf{H}}^{(-\alpha)*}(t, f)$, we have $\Delta_2^{(\alpha)}(t, f) = L_{\mathbf{H}}^{(-\alpha)*}(t, f) - L_{\mathbf{H}}^{(\alpha)*}(t, f)$. Hence, the above bounds follow straightforwardly by applying (12) with $\alpha_1 = -\alpha$, $\alpha_2 = \alpha$. \square

For $|\alpha| \leq 1/2$ one obtains the α -independent (coarser) bounds $2\pi m_{\mathbf{H}}^{(1,1)}$ and $2\pi M_{\mathbf{H}}^{(1,1)}$. For an underspread operator whose spreading function is concentrated along the τ and ν axes so that $m_{\mathbf{H}}^{(1,1)}$ and $M_{\mathbf{H}}^{(1,1)}$ are small, these bounds show that (for $|\alpha|$ not too large) the generalized Weyl symbol of the adjoint of an operator is approximately equal to the complex conjugate of the operator's generalized Weyl symbol,

$$L_{\mathbf{H}^+}^{(\alpha)}(t, f) \approx L_{\mathbf{H}}^{(\alpha)*}(t, f). \quad (14)$$

Due to the similarity of the bounds in Corollary III.2 and Theorem III.1, the above approximation is valid in the same situations where the generalized Weyl symbol is approximately unique. For the special case $\alpha = -1/2$, a result in a similar spirit has been derived by Kohn and Nirenberg³⁷.

An important consequence of Corollary III.2 concerns self-adjoint operators (where $\mathbf{H}^+ = \mathbf{H}$). Whereas in the time-invariant or frequency-invariant case the transfer function of a self-adjoint operator is real-valued, the generalized Weyl symbol of a self-adjoint general operator is not real-valued unless $\alpha = 0$. However, Corollary III.2 implies that the imaginary part of the generalized Weyl symbol of a self-adjoint operator,

$$\mathcal{I}^{(\alpha)}(t, f) \triangleq \frac{1}{2i} [L_{\mathbf{H}}^{(\alpha)}(t, f) - L_{\mathbf{H}}^{(\alpha)*}(t, f)] = \frac{1}{2i} [L_{\mathbf{H}^+}^{(\alpha)}(t, f) - L_{\mathbf{H}}^{(\alpha)*}(t, f)] = \frac{\Delta_2^{(\alpha)}(t, f)}{2i},$$

is bounded as

$$\frac{|\mathcal{I}^{(\alpha)}(t, f)|}{\|S_{\mathbf{H}}\|_1} \leq 2\pi|\alpha| m_{\mathbf{H}}^{(1,1)} \leq \pi m_{\mathbf{H}}^{(1,1)}, \quad \frac{\|\mathcal{I}^{(\alpha)}\|_2}{\|\mathbf{H}\|_2} \leq 2\pi|\alpha| M_{\mathbf{H}}^{(1,1)} \leq \pi M_{\mathbf{H}}^{(1,1)},$$

the coarser bounds being valid for $|\alpha| \leq 1/2$. Hence, the generalized Weyl symbol of an underspread, self-adjoint operator is approximately real-valued even if $\alpha \neq 0$. This result is useful for a discrete-time implementation since here the generalized Weyl symbol with $\alpha = \pm 1/2$ is preferable. Furthermore, the above result shows the approximate real-valuedness of certain time-varying power spectra⁸⁶ in the case $\alpha \neq 0$.

C. Operator Products

The transfer function of the product (composition, series connection) of two time-invariant linear operators \mathbf{H}_1 and \mathbf{H}_2 equals the product of the individual transfer functions, $G_1(f)G_2(f)$. Similarly, the temporal transfer function of a composition of two frequency-invariant linear operators is given by $m_1(t)m_2(t)$. This composition property of conventional transfer functions is the cornerstone of many techniques for signal detection, estimation, filtering, channel equalization, etc.

Contrary to the time-invariant/frequency-invariant case, the generalized Weyl symbol of the product $\mathbf{H}_2\mathbf{H}_1$ of two general linear operators can *not* be obtained by multiplying the individual generalized Weyl symbols of \mathbf{H}_1 and \mathbf{H}_2 . In a signal processing context, this is unfortunate since a multiplicative composition property of the generalized Weyl symbol (a multiplicative symbolic calculus) similar to the time-invariant/frequency-invariant case would allow very simple and intuitive formulations of (statistical) signal processing techniques in nonstationary environments^{42,47–54,87}. In the context of quantum mechanics, such a symbolic calculus—i.e., the approximation of the star product of symbols by the ordinary product—would establish an asymptotic correspondence (in the classical limit $\hbar \rightarrow 0$, where \hbar denotes Planck's constant) between pointwise products and Jordan products as well as between Poisson brackets and commutators¹⁵ (cf. Subsection IV A). Besides the case where \mathbf{H}_1 and \mathbf{H}_2 are either both time-invariant or both frequency-invariant operators (or, in the case $\alpha = 0$, any metaplectic transformation thereof), for Hilbert-Schmidt operators the relation $L_{\mathbf{H}_2\mathbf{H}_1}^{(\alpha)}(t, f) = L_{\mathbf{H}_1}^{(\alpha)}(t, f)L_{\mathbf{H}_2}^{(\alpha)}(t, f)$ is correct only in the following situations:

- For $\alpha = 1/2$, if \mathbf{H}_1 is a linear time-invariant operator with spectral transfer function $G_1(f)$ or \mathbf{H}_2 is a linear frequency-invariant operator with temporal transfer function $m_2(t)$, it can be shown that

$$L_{\mathbf{H}_2\mathbf{H}_1}^{(1/2)}(t, f) = L_{\mathbf{H}_1}^{(1/2)}(t, f)L_{\mathbf{H}_2}^{(1/2)}(t, f) = \begin{cases} G_1(f)L_{\mathbf{H}_2}^{(1/2)}(t, f) & \text{for } \mathbf{H}_1 \text{ time-invariant} \\ L_{\mathbf{H}_1}^{(1/2)}(t, f)m_2(t) & \text{for } \mathbf{H}_2 \text{ frequency-invariant.} \end{cases} \quad (15)$$

- In the dual case $\alpha = -1/2$, if \mathbf{H}_1 is linear frequency-invariant with temporal transfer function $m_1(t)$ or \mathbf{H}_2 is linear time-invariant with spectral transfer function $G_2(f)$, one obtains

$$L_{\mathbf{H}_2\mathbf{H}_1}^{(-1/2)}(t, f) = L_{\mathbf{H}_1}^{(-1/2)}(t, f)L_{\mathbf{H}_2}^{(-1/2)}(t, f) = \begin{cases} m_1(t)L_{\mathbf{H}_2}^{(-1/2)}(t, f) & \text{for } \mathbf{H}_1 \text{ frequency-invariant} \\ L_{\mathbf{H}_1}^{(-1/2)}(t, f)G_2(f) & \text{for } \mathbf{H}_2 \text{ time-invariant.} \end{cases} \quad (16)$$

In general however, the generalized Weyl symbol of operator products $\mathbf{H}_2\mathbf{H}_1$ is described by the so-called *star* (or *twisted*) *product*^{15,31,33,38,40,41,56,69,88,89} $L_{\mathbf{H}_2}^{(\alpha)}\#L_{\mathbf{H}_1}^{(\alpha)}$, which (when generalized to arbitrary α) can be formulated as

$$\begin{aligned} (L_{\mathbf{H}_2}^{(\alpha)}\#L_{\mathbf{H}_1}^{(\alpha)})(t, f) &\triangleq L_{\mathbf{H}_2\mathbf{H}_1}^{(\alpha)}(t, f) = \sum_{k,l=0}^{\infty} c_{kl} \frac{\partial^{k+l}L_{\mathbf{H}_2}^{(\alpha)}(t, f)}{\partial t^l \partial f^k} \frac{\partial^{k+l}L_{\mathbf{H}_1}^{(\alpha)}(t, f)}{\partial t^k \partial f^l} \\ &= L_{\mathbf{H}_2}^{(\alpha)}(t, f)L_{\mathbf{H}_1}^{(\alpha)}(t, f) + \sum_{k+l>0} c_{kl} \frac{\partial^{k+l}L_{\mathbf{H}_2}^{(\alpha)}(t, f)}{\partial t^l \partial f^k} \frac{\partial^{k+l}L_{\mathbf{H}_1}^{(\alpha)}(t, f)}{\partial t^k \partial f^l}, \end{aligned}$$

with $c_{kl} = (-1)^{k+l}(\alpha + 1/2)^k(\alpha - 1/2)^l / [(4\pi^2)^{k+l}k!l!]$. It is thus seen that the difference between $L_{\mathbf{H}_2\mathbf{H}_1}^{(\alpha)}(t, f)$ and $L_{\mathbf{H}_2}^{(\alpha)}(t, f)L_{\mathbf{H}_1}^{(\alpha)}(t, f)$ is essentially determined by the higher-order partial derivatives of the symbols. This difference will be small if these derivatives are small, i.e., if the symbols are smooth functions. The next subsections describe quantitative formulations of this qualitative discussion which are simple and relevant to signal processing applications.

Approximate Multiplicative Symbol Calculus

A positive statement (in the spirit of earlier results obtained in the context of quantization^{14,15,39,56,88}, pseudo-differential operators^{15,37,38,40,41,56}, and transfer function calculus^{42–44}) on the general validity of an approximate multiplicative transfer function (symbolic) calculus is provided by the following theorem.

Theorem III.3. For any two operators \mathbf{H}_1 and \mathbf{H}_2 , the difference $\Delta_3^{(\alpha)}(t, f) \triangleq L_{\mathbf{H}_2\mathbf{H}_1}^{(\alpha)}(t, f) - L_{\mathbf{H}_1}^{(\alpha)}(t, f) L_{\mathbf{H}_2}^{(\alpha)}(t, f)$ is bounded as

$$\frac{|\Delta_3^{(\alpha)}(t, f)|}{\|S_{\mathbf{H}_1}\|_1 \|S_{\mathbf{H}_2}\|_1} \leq 2\pi B_{\mathbf{H}_1, \mathbf{H}_2}^{(\alpha)} \quad \text{with} \quad B_{\mathbf{H}_1, \mathbf{H}_2}^{(\alpha)} \triangleq \left| \alpha + \frac{1}{2} \right| m_{\mathbf{H}_1}^{(0,1)} m_{\mathbf{H}_2}^{(1,0)} + \left| \alpha - \frac{1}{2} \right| m_{\mathbf{H}_1}^{(1,0)} m_{\mathbf{H}_2}^{(0,1)}. \quad (17)$$

Proof. The spreading function of the operator product $\mathbf{H}_2\mathbf{H}_1$ is given by the so-called *twisted convolution*^{15,38,42},

$$S_{\mathbf{H}_2\mathbf{H}_1}^{(\alpha)}(\tau, \nu) = (S_{\mathbf{H}_2}^{(\alpha)} \natural S_{\mathbf{H}_1}^{(\alpha)})(\tau, \nu) \triangleq \int_{\tau'} \int_{\nu'} S_{\mathbf{H}_2}^{(\alpha)}(\tau', \nu') S_{\mathbf{H}_1}^{(\alpha)}(\tau - \tau', \nu - \nu') e^{-i2\pi\phi_\alpha(\tau, \nu, \tau', \nu')} d\tau' d\nu' \quad (18)$$

with $\phi_\alpha(\tau, \nu, \tau', \nu') = (\alpha + 1/2)\tau'(\nu - \nu') + (\alpha - 1/2)(\tau - \tau')\nu'$ (note that the twisted convolution and the star (twisted) product are 2-D Fourier transform pairs). Hence, the Fourier transform of $\Delta_3^{(\alpha)}(t, f)$ is obtained as

$$\begin{aligned} \hat{\Delta}_3^{(\alpha)}(\tau, \nu) &= (S_{\mathbf{H}_2}^{(\alpha)} \natural S_{\mathbf{H}_1}^{(\alpha)})(\tau, \nu) - (S_{\mathbf{H}_2}^{(\alpha)} ** S_{\mathbf{H}_1}^{(\alpha)})(\tau, \nu) \\ &= \int_{\tau'} \int_{\nu'} S_{\mathbf{H}_2}^{(\alpha)}(\tau', \nu') S_{\mathbf{H}_1}^{(\alpha)}(\tau - \tau', \nu - \nu') \left[e^{-i2\pi\phi_\alpha(\tau, \nu, \tau', \nu')} - 1 \right] d\tau' d\nu', \end{aligned}$$

where $**$ denotes ordinary 2-D convolution. Thus, we obtain

$$\begin{aligned} |\Delta_3^{(\alpha)}(t, f)| &\leq \int_{\tau} \int_{\nu} |\hat{\Delta}_3^{(\alpha)}(\tau, \nu)| d\tau d\nu \\ &\leq 2 \int_{\tau} \int_{\nu} \int_{\tau'} \int_{\nu'} |S_{\mathbf{H}_2}^{(\alpha)}(\tau', \nu')| |S_{\mathbf{H}_1}^{(\alpha)}(\tau - \tau', \nu - \nu')| |\sin(\pi\phi_\alpha(\tau, \nu, \tau', \nu'))| d\tau' d\nu' d\tau d\nu. \end{aligned}$$

Substituting $\tau_1 = \tau - \tau'$, $\nu_1 = \nu - \nu'$ and using $|\sin x| \leq |x|$ yields the bound (17) after straightforward manipulations. \square

By virtue of its symmetry with respect to \mathbf{H}_1 and \mathbf{H}_2 , the bound in (17) also applies to the differences $L_{\mathbf{H}_1\mathbf{H}_2}^{(\alpha)}(t, f) - L_{\mathbf{H}_1}^{(\alpha)}(t, f) L_{\mathbf{H}_2}^{(\alpha)}(t, f)$ and $L_{\mathbf{H}_1\star\mathbf{H}_2}^{(\alpha)}(t, f) - L_{\mathbf{H}_1}^{(\alpha)}(t, f) L_{\mathbf{H}_2}^{(\alpha)}(t, f)$, where $\mathbf{H}_1 \star \mathbf{H}_2 = (\mathbf{H}_1\mathbf{H}_2 + \mathbf{H}_2\mathbf{H}_1)/2$ is the (commutative) Jordan product¹⁵ of \mathbf{H}_1 and \mathbf{H}_2 .

The bound shows that if \mathbf{H}_1 and \mathbf{H}_2 are “jointly underspread” such that $m_{\mathbf{H}_1}^{(0,1)} m_{\mathbf{H}_2}^{(1,0)}$ and $m_{\mathbf{H}_1}^{(1,0)} m_{\mathbf{H}_2}^{(0,1)}$ are both small, then we have an approximate multiplicative composition property,

$$L_{\mathbf{H}_2\mathbf{H}_1}^{(\alpha)}(t, f) \approx L_{\mathbf{H}_1}^{(\alpha)}(t, f) L_{\mathbf{H}_2}^{(\alpha)}(t, f). \quad (19)$$

This approximate relation provides a basis for several signal processing methods for nonstationary environments recently proposed for a variety of problems like equalization of mobile radio channels^{48,49}, estimation of nonstationary random signals corrupted by noise^{42,47,50,51,53,87}, and detection or classification of nonstationary signals^{52,53,87}. Furthermore, the above approximation can be used to show that two important time-varying power spectra of nonstationary random processes—the Wigner-Ville spectrum^{86,90} and the (generalized) evolutionary spectrum^{54,91}—are basically equivalent^{42,54,55,87}.

We note that (19) implies $L_{\mathbf{H}_2\mathbf{H}_1}^{(\alpha)}(t, f) \approx L_{\mathbf{H}_1\mathbf{H}_2}^{(\alpha)}(t, f)$, which suggests that two jointly underspread operators are approximately commuting, i.e., $\mathbf{H}_2\mathbf{H}_1 \approx \mathbf{H}_1\mathbf{H}_2$. This issue will be investigated further in Subsection IV A.

In the following, we briefly discuss the approximation (19) for the cases $\alpha = \pm 1/2$. The case $\alpha = 0$ deserves special attention and will be considered in more detail afterwards.

For $\alpha = 1/2$, $B_{\mathbf{H}_1, \mathbf{H}_2}^{(\alpha)}$ simplifies to $B_{\mathbf{H}_1, \mathbf{H}_2}^{(1/2)} = m_{\mathbf{H}_1}^{(0,1)} m_{\mathbf{H}_2}^{(1,0)}$, which will be small if $|S_{\mathbf{H}_1}(\tau, \nu)|$ is concentrated along the τ axis and/or $|S_{\mathbf{H}_2}(\tau, \nu)|$ is concentrated along the ν axis. Thus, for $\alpha = 1/2$ the approximation (19) is good when \mathbf{H}_1 is a quasi-time-invariant system and/or \mathbf{H}_2 is a quasi-frequency-invariant system (note the consistency with the exact multiplicative calculus in (15)).

For $\alpha = -1/2$, we have $B_{\mathbf{H}_1, \mathbf{H}_2}^{(-1/2)} = m_{\mathbf{H}_1}^{(1,0)} m_{\mathbf{H}_2}^{(0,1)}$. This will be small if $|S_{\mathbf{H}_1}(\tau, \nu)|$ is concentrated along the ν axis and/or $|S_{\mathbf{H}_2}(\tau, \nu)|$ is concentrated along the τ axis, i.e., if \mathbf{H}_1 is quasi-frequency-invariant and/or \mathbf{H}_2 is quasi-time-invariant (note the consistency with (16)).

For both $\alpha = 1/2$ and $\alpha = -1/2$, the bound is large, and thus the approximation (19) is poor, if the operators' spreading functions are oriented in oblique directions. For any α with $|\alpha| \leq 1/2$, it follows from the definition

of $B_{\mathbf{H}_1, \mathbf{H}_2}^{(\alpha)}$ in (17) that $B_{\mathbf{H}_1, \mathbf{H}_2}^{(\alpha)}$ is a convex combination of the two extreme cases $B_{\mathbf{H}_1, \mathbf{H}_2}^{(1/2)} = m_{\mathbf{H}_1}^{(0,1)} m_{\mathbf{H}_2}^{(1,0)}$ and $B_{\mathbf{H}_1, \mathbf{H}_2}^{(-1/2)} = m_{\mathbf{H}_1}^{(1,0)} m_{\mathbf{H}_2}^{(0,1)}$, i.e.,

$$B_{\mathbf{H}_1, \mathbf{H}_2}^{(\alpha)} = \gamma B_{\mathbf{H}_1, \mathbf{H}_2}^{(1/2)} + (1 - \gamma) B_{\mathbf{H}_1, \mathbf{H}_2}^{(-1/2)} \quad \text{with } 0 \leq \gamma = \alpha + \frac{1}{2} \leq 1.$$

In particular, this implies

$$\min \{B_{\mathbf{H}_1, \mathbf{H}_2}^{(1/2)}, B_{\mathbf{H}_1, \mathbf{H}_2}^{(-1/2)}\} \leq B_{\mathbf{H}_1, \mathbf{H}_2}^{(\alpha)} \leq \max \{B_{\mathbf{H}_1, \mathbf{H}_2}^{(1/2)}, B_{\mathbf{H}_1, \mathbf{H}_2}^{(-1/2)}\}, \quad |\alpha| \leq \frac{1}{2}.$$

Case $\alpha = 0$

The case $\alpha = 0$ is of particular importance. Here, $B_{\mathbf{H}_1, \mathbf{H}_2}^{(\alpha)}$ simplifies to

$$B_{\mathbf{H}_1, \mathbf{H}_2}^{(0)} = \frac{1}{2} \left[m_{\mathbf{H}_1}^{(0,1)} m_{\mathbf{H}_2}^{(1,0)} + m_{\mathbf{H}_1}^{(1,0)} m_{\mathbf{H}_2}^{(0,1)} \right],$$

which is symmetric with respect to \mathbf{H}_1 and \mathbf{H}_2 . Thus, the approximation (19) will be good if both $m_{\mathbf{H}_1}^{(0,1)} m_{\mathbf{H}_2}^{(1,0)}$ and $m_{\mathbf{H}_1}^{(1,0)} m_{\mathbf{H}_2}^{(0,1)}$ are small. This requires that the spreading functions of \mathbf{H}_1 and \mathbf{H}_2 are both concentrated about the origin of the (τ, ν) plane, with *similar* orientation parallel to the τ or ν axis. However, this requirement is relaxed by a refinement of the bound $2\pi B_{\mathbf{H}_1, \mathbf{H}_2}^{(0)}$ that is stated in the next theorem. This improved bound can be attributed to the covariance of the Weyl symbol $L_{\mathbf{H}}^{(0)}(t, f)$ to metaplectic transformations $\mathbf{U} \in \mathcal{M}$ (cf. (4)).

Theorem III.4. *For any two operators \mathbf{H}_1 and \mathbf{H}_2 , the difference $\Delta_3^{(0)}(t, f) = L_{\mathbf{H}_2 \mathbf{H}_1}^{(0)}(t, f) - L_{\mathbf{H}_1}^{(0)}(t, f) L_{\mathbf{H}_2}^{(0)}(t, f)$ obeys the bound (that is generally tighter than (17))*

$$\frac{|\Delta_3^{(0)}(t, f)|}{\|S_{\mathbf{H}_1}\|_1 \|S_{\mathbf{H}_2}\|_1} \leq 2\pi \min_{\mathbf{U} \in \mathcal{M}} B_{\mathbf{U} \mathbf{H}_1 \mathbf{U}^+, \mathbf{U} \mathbf{H}_2 \mathbf{U}^+}^{(0)}. \quad (20)$$

Proof. Specializing the proof of Theorem III.3 to $\alpha = 0$, it is seen that

$$|\Delta_3^{(0)}(t, f)| \leq \pi \int_{\tau_1} \int_{\nu_1} \int_{\tau_2} \int_{\nu_2} |S_{\mathbf{H}_2}(\tau_1, \nu_1)| |S_{\mathbf{H}_1}(\tau_2, \nu_2)| |\tau_1 \nu_2 - \tau_2 \nu_1| d\tau_1 d\nu_1 d\tau_2 d\nu_2. \quad (21)$$

Performing a symplectic coordinate transform $\begin{pmatrix} \tau_i \\ \nu_i \end{pmatrix} = \mathcal{A} \begin{pmatrix} \tau'_i \\ \nu'_i \end{pmatrix}$, where $\mathcal{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det \mathcal{A} = ad - bc = 1$, and using the invariance of the symplectic form^{15,73,75}, $\tau_1 \nu_2 - \tau_2 \nu_1 = \tau'_1 \nu'_2 - \tau'_2 \nu'_1$, the right-hand side of (21) becomes

$$\pi \int_{\tau'_1} \int_{\nu'_1} \int_{\tau'_2} \int_{\nu'_2} |S_{\mathbf{H}_2}(a\tau'_1 + b\nu'_1, c\tau'_1 + d\nu'_1)| |S_{\mathbf{H}_1}(a\tau'_2 + b\nu'_2, c\tau'_2 + d\nu'_2)| |\tau'_1 \nu'_2 - \tau'_2 \nu'_1| d\tau'_1 d\nu'_1 d\tau'_2 d\nu'_2.$$

Let $\tilde{\mathbf{H}}_i = \mathbf{U} \mathbf{H}_i \mathbf{U}^+$ where $\mathbf{U} \in \mathcal{M}$ is the unitary operator corresponding to \mathcal{A} via the metaplectic representation¹⁵. Using the covariance property (8), the above expression becomes

$$\pi \int_{\tau'_1} \int_{\nu'_1} \int_{\tau'_2} \int_{\nu'_2} |S_{\tilde{\mathbf{H}}_2}(\tau'_1, \nu'_1)| |S_{\tilde{\mathbf{H}}_1}(\tau'_2, \nu'_2)| |\tau'_1 \nu'_2 - \tau'_2 \nu'_1| d\tau'_1 d\nu'_1 d\tau'_2 d\nu'_2.$$

Inserting $|\tau'_1 \nu'_2 - \tau'_2 \nu'_1| \leq |\tau'_1 \nu'_2| + |\tau'_2 \nu'_1|$ and using $\|\tilde{\mathbf{H}}_i\|_1 = \|\mathbf{H}_i\|_1$ (which holds since the spreading function magnitude of $\tilde{\mathbf{H}}_i$ is obtained from that of \mathbf{H}_i via an area-preserving coordinate transform), we obtain

$$|\Delta_3^{(0)}(t, f)| \leq \pi \|S_{\mathbf{H}_1}\|_1 \|S_{\mathbf{H}_2}\|_1 \left[m_{\tilde{\mathbf{H}}_1}^{(0,1)} m_{\tilde{\mathbf{H}}_2}^{(1,0)} + m_{\tilde{\mathbf{H}}_1}^{(1,0)} m_{\tilde{\mathbf{H}}_2}^{(0,1)} \right].$$

As this bound is valid for all $\mathbf{U} \in \mathcal{M}$, we finally obtain the bound (20). \square

Since the symplectic group contains time-frequency rotations and time-frequency shearings, this theorem shows that $\Delta_3^{(0)}(t, f)$ may be small even if the spreading functions of \mathbf{H}_1 and \mathbf{H}_2 are oriented in (similar) *oblique* directions. This is not true for $\alpha \neq 0$, which once again shows the exceptional position of the Weyl symbol.

Composition of \mathbf{H} with \mathbf{H}^+

In some applications, e.g. for an innovations system representation of random processes⁵⁴ or for a quadratic time-frequency analysis of linear operators⁹², the composition of \mathbf{H} with \mathbf{H}^+ is of importance (cf. also the proofs of Theorems III.6 and III.12). For time-invariant or frequency-invariant operators, the transfer function of $\mathbf{H}^+\mathbf{H}$ is $|G(f)|^2$ or $|m(t)|^2$, respectively. For general operators, the following result can be obtained from Theorem III.3 and Corollary III.2.

Corollary III.5. *For any operator \mathbf{H} , the difference $\Delta_4^{(\alpha)}(t, f) \triangleq L_{\mathbf{H}^+\mathbf{H}}^{(\alpha)}(t, f) - |L_{\mathbf{H}}^{(\alpha)}(t, f)|^2$ is bounded as*

$$\frac{|\Delta_4^{(\alpha)}(t, f)|}{\|S_{\mathbf{H}}\|_1^2} \leq 2\pi C_{\mathbf{H}}^{(\alpha)} \quad \text{with} \quad C_{\mathbf{H}}^{(\alpha)} \triangleq c_{\alpha} m_{\mathbf{H}}^{(0,1)} m_{\mathbf{H}}^{(1,0)} + 2|\alpha| m_{\mathbf{H}}^{(1,1)}, \quad (22)$$

where $c_{\alpha} = |\alpha + 1/2| + |\alpha - 1/2|$.

Proof. Subtracting and adding $L_{\mathbf{H}}^{(\alpha)}(t, f)L_{\mathbf{H}^+}^{(\alpha)}(t, f)$ from/to $\Delta_4^{(\alpha)}(t, f)$, we obtain

$$\begin{aligned} |\Delta_4^{(\alpha)}(t, f)| &\leq \left| L_{\mathbf{H}^+\mathbf{H}}^{(\alpha)}(t, f) - L_{\mathbf{H}}^{(\alpha)}(t, f)L_{\mathbf{H}^+}^{(\alpha)}(t, f) \right| + \left| L_{\mathbf{H}^+}^{(\alpha)}(t, f) - L_{\mathbf{H}}^{(\alpha)*}(t, f) \right| \left| L_{\mathbf{H}}^{(\alpha)}(t, f) \right| \\ &\leq 2\pi c_{\alpha} \|S_{\mathbf{H}}\|_1^2 m_{\mathbf{H}}^{(1,0)} m_{\mathbf{H}}^{(0,1)} + 4\pi|\alpha| m_{\mathbf{H}}^{(1,1)} \|S_{\mathbf{H}}\|_1 \left| L_{\mathbf{H}}^{(\alpha)}(t, f) \right| \end{aligned}$$

where we used (17) with $m_{\mathbf{H}^+}^{(k,l)} = m_{\mathbf{H}}^{(k,l)}$ and $\|S_{\mathbf{H}^+}\|_1 = \|S_{\mathbf{H}}\|_1$ as well as (13). From this, (22) follows with $|L_{\mathbf{H}}^{(\alpha)}(t, f)| \leq \|S_{\mathbf{H}}\|_1$. \square

Hence, for an underspread operator with small $m_{\mathbf{H}}^{(0,1)} m_{\mathbf{H}}^{(1,0)}$ and small $m_{\mathbf{H}}^{(1,1)}$, we have

$$L_{\mathbf{H}^+\mathbf{H}}^{(\alpha)}(t, f) \approx |L_{\mathbf{H}}^{(\alpha)}(t, f)|^2,$$

i.e., “squaring” the operator is approximately equivalent to squaring its generalized Weyl symbol. We note that the bound (22), and thus the above approximation, remain valid if $\mathbf{H}^+\mathbf{H}$ is replaced by $\mathbf{H}\mathbf{H}^+$ or by $\mathbf{H} \star \mathbf{H}^+ = (\mathbf{H}\mathbf{H}^+ + \mathbf{H}^+\mathbf{H})/2$.

For $|\alpha| \leq 1/2$, we have $c_{\alpha} = 1$ and thus $C_{\mathbf{H}}^{(\alpha)} \leq m_{\mathbf{H}}^{(1,0)} m_{\mathbf{H}}^{(0,1)} + m_{\mathbf{H}}^{(1,1)}$. The bound $C_{\mathbf{H}}^{(\alpha)}$ is tightest for $\alpha = 0$, in which case $C_{\mathbf{H}}^{(0)} = m_{\mathbf{H}}^{(0,1)} m_{\mathbf{H}}^{(1,0)}$. This is due to the fact that the Weyl symbol of the adjoint is *exactly* obtained by complex conjugation. Furthermore, for $\alpha = 0$ the above result can be refined similarly to Theorem III.4, yielding the tighter bound

$$\frac{|\Delta_4^{(0)}(t, f)|}{\|S_{\mathbf{H}}\|_1^2} \leq 2\pi \min_{\mathbf{U} \in \mathcal{M}} C_{\mathbf{U}\mathbf{H}\mathbf{U}^+}^{(0)} = 2\pi \min_{\mathbf{U} \in \mathcal{M}} m_{\mathbf{U}\mathbf{H}\mathbf{U}^+}^{(0,1)} m_{\mathbf{U}\mathbf{H}\mathbf{U}^+}^{(1,0)}. \quad (23)$$

D. Approximate Eigenvalues and Eigenfunctions

The response of a time-invariant system to a complex sinusoid $e^{i2\pi f_0 t}$ —i.e., a signal with perfect frequency concentration—is $G(f_0) e^{i2\pi f_0 t}$. Thus, the complex sinusoids $e^{i2\pi f_0 t}$ are the generalized⁶⁰ eigenfunctions of any time-invariant system, with $G(f_0)$ the associated generalized eigenvalues. Similarly, the response of a frequency-invariant system to a Dirac impulse $\delta(t - t_0)$ —i.e., a signal perfectly localized in time—is given by $m(t_0) \delta(t - t_0)$, and hence the Dirac impulses $\delta(t - t_0)$ are the generalized eigenfunctions with $m(t_0)$ the associated generalized eigenvalues. Note that in the time-invariant and frequency-invariant cases the eigenfunctions are highly structured since they are obtained by frequency shifts of the unity function 1 and by time shifts of the Dirac impulse $\delta(t)$, respectively; moreover, the eigenvalues are given by the values of the transfer function.

On the other hand, the eigenfunctions (singular functions) of general linear operators are not localized and structured in any sense, and the eigenvalues (singular values) are not given by the transfer function values. However, we will now show that *underspread* operators have a well-structured set of time-frequency-localized “approximate eigenfunctions,” with the associated “approximate eigenvalues” given by the values of the generalized Weyl symbol. Note that it is not necessary to consider approximate singular functions since underspread operators are approximately normal—see Subsection IV B.

Let $s(t)$ be a normalized function that is well concentrated about the origin of the time-frequency plane (e.g., a Gaussian function). We consider the family of functions $s_{t_0, f_0}(t) = s(t - t_0) e^{i2\pi f_0 t}$ obtained by time-frequency-shifting $s(t)$ to the time-frequency point (t_0, f_0) . By construction, $s_{t_0, f_0}(t)$ is then well time-frequency-concentrated about (t_0, f_0) . The practical usefulness of the set $\{s_{t_0, f_0}(t)\}$ stems from the fact that it is highly structured and parameterized by physically meaningful variables. Such time-delayed and frequency-shifted signal sets are particularly appealing in communication applications, for example in mobile radio and digital broadcasting^{93–96}. We shall now state conditions under which the response of a linear operator \mathbf{H} to $s_{t_0, f_0}(t)$ is approximately $L_{\mathbf{H}}^{(\alpha)}(t_0, f_0) s_{t_0, f_0}(t)$, which implies that $s_{t_0, f_0}(t)$ is an “approximate eigenfunction” of \mathbf{H} with $L_{\mathbf{H}}^{(\alpha)}(t_0, f_0)$ the associated “approximate eigenvalue.” The next theorem is essentially an adaptation and extension of an existing result on approximate eigenpairs^{42,95}. (Approximate eigenpairs and approximate operator diagonalization have previously been considered also at a mathematically more sophisticated level^{14,15,17,97–99}.) In the following,

$$A_s^{(\alpha)}(\tau, \nu) \triangleq \int_t s\left(t + \left(\frac{1}{2} - \alpha\right)\tau\right) s^*\left(t - \left(\frac{1}{2} + \alpha\right)\tau\right) e^{-i2\pi\nu t} dt$$

denotes the (generalized) ambiguity function^{100,101} of the function $s(t)$, which equals the spreading function of the rank-one operator with kernel $h(t, t') = s(t) s^*(t')$. We note that $A_s^{(\alpha)}(0, 0) = \|s\|_2^2 = 1$.

Theorem III.6. *For any operator \mathbf{H} , any time-frequency point (t_0, f_0) , and any normalized function $s(t)$ (i.e., $\|s\|_2 = 1$), the difference $\Delta_5^{(\alpha)}(t) \triangleq (\mathbf{H} s_{t_0, f_0})(t) - L_{\mathbf{H}}^{(\alpha)}(t_0, f_0) s_{t_0, f_0}(t)$ is bounded as*

$$\frac{\|\Delta_5^{(\alpha)}\|_2}{\|S_{\mathbf{H}}\|_1} \leq D_{\mathbf{H}, s}^{(\alpha)} \quad \text{with} \quad D_{\mathbf{H}, s}^{(\alpha)} \triangleq \sqrt{2\pi C_{\mathbf{H}}^{(\alpha)} + m_{\mathbf{H}+\mathbf{H}}^{(\phi_s)} + 2m_{\mathbf{H}}^{(\phi_s)}}, \quad (24)$$

with the weighting function $\phi_s(\tau, \nu) = |1 - A_s^{(\alpha)}(\tau, \nu)|$ and $C_{\mathbf{H}}^{(\alpha)}$ as defined in (22).

Proof. First, we develop the square of $\|\Delta_5^{(\alpha)}\|_2$ as

$$\begin{aligned} \|\Delta_5^{(\alpha)}\|_2^2 &= \|\mathbf{H} s_{t_0, f_0}\|_2^2 - 2 \operatorname{Re} \left\{ L_{\mathbf{H}}^{(\alpha)*}(t_0, f_0) \langle \mathbf{H} s_{t_0, f_0}, s_{t_0, f_0} \rangle \right\} + |L_{\mathbf{H}}^{(\alpha)}(t_0, f_0)|^2 \|s_{t_0, f_0}\|_2^2 \\ &= \|\mathbf{H} s_{t_0, f_0}\|_2^2 - 2 \operatorname{Re} \left\{ L_{\mathbf{H}}^{(\alpha)*}(t_0, f_0) \left[\langle \mathbf{H} s_{t_0, f_0}, s_{t_0, f_0} \rangle - L_{\mathbf{H}}^{(\alpha)}(t_0, f_0) \right] \right\} - |L_{\mathbf{H}}^{(\alpha)}(t_0, f_0)|^2 \\ &\leq \left| \|\mathbf{H} s_{t_0, f_0}\|_2^2 - |L_{\mathbf{H}}^{(\alpha)}(t_0, f_0)|^2 \right| + 2 |L_{\mathbf{H}}^{(\alpha)}(t_0, f_0)| \left| \langle \mathbf{H} s_{t_0, f_0}, s_{t_0, f_0} \rangle - L_{\mathbf{H}}^{(\alpha)}(t_0, f_0) \right|, \end{aligned} \quad (25)$$

where we used $\|s_{t_0, f_0}\|_2 = \|s\|_2 = 1$. Using

$$\langle \mathbf{G} s_{t_0, f_0}, s_{t_0, f_0} \rangle = \int_{\tau} \int_{\nu} S_{\mathbf{G}}^{(\alpha)}(\tau, \nu) A_s^{(\alpha)*}(\tau, \nu) e^{i2\pi(\nu t_0 - \tau f_0)} d\tau d\nu \quad (26)$$

with $\mathbf{G} = \mathbf{H} + \mathbf{H}$ and recalling the definition of $\Delta_4^{(\alpha)}(t, f)$ in Corollary III.5, the first term in (25) becomes

$$\begin{aligned} \left| \|\mathbf{H} s_{t_0, f_0}\|_2^2 - |L_{\mathbf{H}}^{(\alpha)}(t_0, f_0)|^2 \right| &= \left| \langle \mathbf{H} + \mathbf{H} s_{t_0, f_0}, s_{t_0, f_0} \rangle - L_{\mathbf{H}+\mathbf{H}}^{(\alpha)}(t_0, f_0) + L_{\mathbf{H}+\mathbf{H}}^{(\alpha)}(t_0, f_0) - |L_{\mathbf{H}}^{(\alpha)}(t_0, f_0)|^2 \right| \\ &= \left| \int_{\tau} \int_{\nu} S_{\mathbf{H}+\mathbf{H}}^{(\alpha)}(\tau, \nu) \left[A_s^{(\alpha)*}(\tau, \nu) - 1 \right] e^{i2\pi(\nu t_0 - \tau f_0)} d\tau d\nu + \Delta_4^{(\alpha)}(t_0, f_0) \right| \\ &\leq \int_{\tau} \int_{\nu} |S_{\mathbf{H}+\mathbf{H}}(\tau, \nu)| \left| 1 - A_s^{(\alpha)}(\tau, \nu) \right| d\tau d\nu + |\Delta_4^{(\alpha)}(t_0, f_0)| \\ &\leq \|S_{\mathbf{H}+\mathbf{H}}\|_1 m_{\mathbf{H}+\mathbf{H}}^{(\phi_s)} + \|S_{\mathbf{H}}\|_1^2 2\pi C_{\mathbf{H}}^{(\alpha)} \leq \|S_{\mathbf{H}}\|_1^2 \left[m_{\mathbf{H}+\mathbf{H}}^{(\phi_s)} + 2\pi C_{\mathbf{H}}^{(\alpha)} \right] \end{aligned}$$

where we used (22) and Young’s inequality for twisted convolution¹⁵, $\|S_{\mathbf{H}+\mathbf{H}}\|_1 \leq \|S_{\mathbf{H}}\|_1^2$. In a similar manner, the factor of the second term in (25) can be developed and bounded as

$$\left| \langle \mathbf{H} s_{t_0, f_0}, s_{t_0, f_0} \rangle - L_{\mathbf{H}}^{(\alpha)}(t_0, f_0) \right| \leq \int_{\tau} \int_{\nu} |S_{\mathbf{H}}(\tau, \nu)| \left| 1 - A_s^{(\alpha)}(\tau, \nu) \right| d\tau d\nu = \|S_{\mathbf{H}}\|_1 m_{\mathbf{H}}^{(\phi_s)}.$$

Inserting these two bounds into (25) and using $|L_{\mathbf{H}}^{(\alpha)}(t_0, f_0)| \leq \|S_{\mathbf{H}}\|_1$ yields the bound (24). \square

For an underspread operator \mathbf{H} where $D_{\mathbf{H},s}^{(\alpha)} = \sqrt{2\pi C_{\mathbf{H}}^{(\alpha)} + m_{\mathbf{H}+\mathbf{H}}^{(\phi_s)} + 2m_{\mathbf{H}}^{(\phi_s)}}$ can be made small, we thus have the “approximate eigenvalue relation”

$$(\mathbf{H} s_{t_0, f_0})(t) \approx L_{\mathbf{H}}^{(\alpha)}(t_0, f_0) s_{t_0, f_0}(t),$$

which implies that properly time-frequency-localized functions are approximate eigenfunctions and the generalized Weyl symbol $L_{\mathbf{H}}^{(\alpha)}(t, f)$ is an approximate eigenvalue distribution over the time-frequency plane. In particular, small $m_{\mathbf{H}}^{(\phi_s)}$ and $m_{\mathbf{H}+\mathbf{H}}^{(\phi_s)}$ requires that, on the effective supports of $|S_{\mathbf{H}}(\tau, \nu)|$ and of $|S_{\mathbf{H}+\mathbf{H}}(\tau, \nu)|$, there is $\phi_s(\tau, \nu) \approx 0$ (recall the definition (9)) and thus $A_s^{(\alpha)}(\tau, \nu) \approx A_s^{(\alpha)}(0, 0) \equiv 1$. The latter condition can be satisfied only if these effective supports are small, i.e., if \mathbf{H} is underspread⁵. Moreover, $A_s^{(\alpha)}(\tau, \nu) \approx A_s^{(\alpha)}(0, 0)$ around the origin implies that the second-order derivatives of $A_s^{(\alpha)}(\tau, \nu)$ with respect to τ and ν are small at the origin. Since these second-order derivatives correspond to the effective duration and bandwidth^{102,103} of $s(t)$, it is seen that the function $s(t)$ needs to have good time-frequency concentration.

The bound $D_{\mathbf{H},s}^{(\alpha)}$ is tightest for $\alpha = 0$ since here $C_{\mathbf{H}}^{(\alpha)}$ is smallest. Furthermore, $C_{\mathbf{H}}^{(0)}$ can be replaced by $\min_{\mathbf{U} \in \mathcal{M}} C_{\mathbf{U}\mathbf{H}\mathbf{U}^+}^{(0)}$ which allows oblique orientation of the spreading function of \mathbf{H} . We note that $m_{\mathbf{H}}^{(\phi_s)}$ and $m_{\mathbf{H}+\mathbf{H}}^{(\phi_s)}$ can be adapted to an oblique orientation of the spreading function by proper choice of the function $s(t)$.

E. Input-Output Relation for Deterministic Signals

For time-invariant systems, due to the fact that the complex sinusoids are eigenfunctions, the Fourier transform of the output signal $(\mathbf{H}x)(t)$ is given by $G(f)X(f)$. Similarly, for frequency-invariant systems $(\mathbf{H}x)(t)$ equals $m(t)x(t)$. In the case of general operators, one may desire a similar “input-output relation” stating that a suitably defined time-frequency representation of $(\mathbf{H}x)(t)$ equals the time-frequency representation of $x(t)$ multiplied by the time-frequency transfer function $L_{\mathbf{H}}^{(\alpha)}(t, f)$. To derive such an input-output relation, we note that the input signal $x(t)$ can be expanded^{15,17,21–23} into the set of time-frequency-shifted versions $s_{t_0, f_0}(t) = s(t - t_0) e^{i2\pi f_0 t}$ of a normalized function $s(t)$,

$$x(t) = \int_{t_0} \int_{f_0} \text{STFT}_x^{(s)}(t_0, f_0) s_{t_0, f_0}(t) dt_0 df_0, \quad (27)$$

where

$$\text{STFT}_x^{(s)}(t, f) = \int_{t'} x(t') s^*(t' - t) e^{-i2\pi f t'} dt' = \langle x, s_{t, f} \rangle \quad (28)$$

is the *short-time Fourier transform*^{9,21,22,24} of the signal $x(t)$ using $s(t)$ as analysis window. For Gaussian $s(t)$, the short-time Fourier transform is equivalent to the coherent states expansions of quantum physics^{15–18,104}. In what follows, we will adopt the short-time Fourier transform as time-frequency signal representation. Assuming $s(t)$ to have good time-frequency concentration and \mathbf{H} to be underspread, we know from Subsection III D that the $s_{t_0, f_0}(t)$ are approximate eigenfunctions, i.e., $(\mathbf{H} s_{t_0, f_0})(t) \approx L_{\mathbf{H}}^{(\alpha)}(t_0, f_0) s_{t_0, f_0}(t)$. Hence, applying \mathbf{H} to both sides of (27) we obtain

$$\begin{aligned} (\mathbf{H}x)(t) &= \int_{t_0} \int_{f_0} \text{STFT}_x^{(s)}(t_0, f_0) (\mathbf{H} s_{t_0, f_0})(t) dt_0 df_0 \\ &\approx \int_{t_0} \int_{f_0} \text{STFT}_x^{(s)}(t_0, f_0) L_{\mathbf{H}}^{(\alpha)}(t_0, f_0) s_{t_0, f_0}(t) dt_0 df_0. \end{aligned}$$

Comparing with (27) (with $x(t)$ replaced by $(\mathbf{H}x)(t)$), this suggests that $\text{STFT}_{\mathbf{H}x}^{(s)}(t, f) \approx \text{STFT}_x^{(s)}(t, f) L_{\mathbf{H}}^{(\alpha)}(t, f)$. Hence, the generalized Weyl symbol can be interpreted as a weighting function in the time-frequency (short-time

⁵Recall from (18) that $S_{\mathbf{H}+\mathbf{H}}^{(\alpha)}(\tau, \nu) = (S_{\mathbf{H}}^{(\alpha)} \natural S_{\mathbf{H}}^{(\alpha)})(\tau, \nu)$, and note that this twisted convolution enlarges the effective support of $S_{\mathbf{H}}^{(\alpha)}(\tau, \nu)$ maximally by a factor of four^{42,44}.

Fourier transform) domain, thus motivating a design and implementation of time-varying filters based on the short-time Fourier transform (see Subsection III F). The next theorem gives a bound on the quality of the above approximation.

Theorem III.7. *For any operator \mathbf{H} , any signal $x(t)$, and any normalized function $s(t)$, the difference $\Delta_6^{(\alpha)}(t, f) \triangleq \text{STFT}_{\mathbf{H}x}^{(s)}(t, f) - L_{\mathbf{H}}^{(\alpha)}(t, f) \text{STFT}_x^{(s)}(t, f)$ is bounded as*

$$\frac{|\Delta_6^{(\alpha)}(t, f)|}{\|\mathbf{S}_{\mathbf{H}}\|_1 \|x\|_2} \leq D_{\mathbf{H}^+, s}^{(\alpha)} + 4\pi|\alpha| m_{\mathbf{H}}^{(1,1)}, \quad \frac{\|\Delta_6^{(\alpha)}\|_2}{\|\mathbf{H}\|_2 \|x\|_2} \leq \sqrt{2} M_{\mathbf{H}}^{(\phi'_s)} + 4\pi|\alpha| M_{\mathbf{H}}^{(1,1)}, \quad (29)$$

with the weighting functions $\phi_s(\tau, \nu) = |1 - A_s^{(\alpha)}(\tau, \nu)|$ (used in $D_{\mathbf{H}^+, s}^{(\alpha)}$, see (24)) and $\phi'_s(\tau, \nu) = \sqrt{1 - \text{Re}\{A_s^{(\alpha)}(\tau, \nu)\}}$ (used in $M_{\mathbf{H}}^{(\phi'_s)}$).

Proof. We have

$$\begin{aligned} \Delta_6^{(\alpha)}(t, f) &= \langle \mathbf{H}x, s_{t,f} \rangle - L_{\mathbf{H}}^{(\alpha)}(t, f) \langle x, s_{t,f} \rangle = \left\langle x, \left[\mathbf{H}^+ - L_{\mathbf{H}}^{(\alpha)*}(t, f) \mathbf{I} \right] s_{t,f} \right\rangle \\ &= \left\langle x, \left[\mathbf{H}^+ - L_{\mathbf{H}^+}^{(\alpha)}(t, f) \mathbf{I} + L_{\mathbf{H}^+}^{(\alpha)}(t, f) \mathbf{I} - L_{\mathbf{H}}^{(\alpha)*}(t, f) \mathbf{I} \right] s_{t,f} \right\rangle = \Delta_A^{(\alpha)}(t, f) + \Delta_B^{(\alpha)}(t, f) \end{aligned} \quad (30)$$

with

$$\Delta_A^{(\alpha)}(t, f) \triangleq \langle x, [\mathbf{H}^+ - L_{\mathbf{H}^+}^{(\alpha)}(t, f) \mathbf{I}] s_{t,f} \rangle, \quad \Delta_B^{(\alpha)}(t, f) \triangleq [L_{\mathbf{H}^+}^{(\alpha)}(t, f) - L_{\mathbf{H}}^{(\alpha)*}(t, f)]^* \langle x, s_{t,f} \rangle.$$

Using Schwarz' inequality, (24), (13), and $\|s_{t,f}\|_2 = \|s\|_2 = 1$, we obtain the bounds

$$|\Delta_A^{(\alpha)}(t, f)| \leq \|x\|_2 \|\mathbf{H}^+ s_{t,f} - L_{\mathbf{H}^+}^{(\alpha)}(t, f) s_{t,f}\|_2 \leq \|x\|_2 \|\mathbf{S}_{\mathbf{H}^+}\|_1 D_{\mathbf{H}^+, s}^{(\alpha)} = \|x\|_2 \|\mathbf{S}_{\mathbf{H}}\|_1 D_{\mathbf{H}^+, s}^{(\alpha)} \quad (31)$$

$$|\Delta_B^{(\alpha)}(t, f)| \leq |L_{\mathbf{H}^+}^{(\alpha)}(t, f) - L_{\mathbf{H}}^{(\alpha)*}(t, f)| \|x\|_2 \|s_{t,f}\|_2 \leq \|x\|_2 \|\mathbf{S}_{\mathbf{H}}\|_1 4\pi|\alpha| m_{\mathbf{H}}^{(1,1)}. \quad (32)$$

The first (L_∞) bound in (29) then follows upon inserting (31) and (32) into $|\Delta_6^{(\alpha)}(t, f)| \leq |\Delta_A^{(\alpha)}(t, f)| + |\Delta_B^{(\alpha)}(t, f)|$ (cf. (30)).

To derive the second (L_2) bound in (29), we note that

$$\begin{aligned} \|\Delta_A^{(\alpha)}\|_2^2 &= \int_t \int_f |\langle x, \mathbf{H}^+ s_{t,f} - L_{\mathbf{H}^+}^{(\alpha)}(t, f) s_{t,f} \rangle|^2 dt df \leq \int_t \int_f \|x\|_2^2 \|\mathbf{H}^+ s_{t,f} - L_{\mathbf{H}^+}^{(\alpha)}(t, f) s_{t,f}\|_2^2 dt df \\ &= \|x\|_2^2 \int_t \int_f \left[\|\mathbf{H}^+ s_{t,f}\|_2^2 - 2 \text{Re}\{L_{\mathbf{H}^+}^{(\alpha)*}(t, f) \langle \mathbf{H}^+ s_{t,f}, s_{t,f} \rangle\} + |L_{\mathbf{H}^+}^{(\alpha)}(t, f)|^2 \|s_{t,f}\|_2^2 \right] dt df. \end{aligned} \quad (33)$$

Using (26), it can be shown that

$$\int_t \int_f \|\mathbf{H}^+ s_{t,f}\|_2^2 dt df = \int_t \int_f \langle \mathbf{H} \mathbf{H}^+ s_{t,f}, s_{t,f} \rangle dt df = S_{\mathbf{H} \mathbf{H}^+}^{(\alpha)}(0, 0) = \text{Tr}\{\mathbf{H} \mathbf{H}^+\} = \|\mathbf{S}_{\mathbf{H}}\|_2^2$$

and

$$\int_t \int_f L_{\mathbf{H}^+}^{(\alpha)*}(t, f) \langle \mathbf{H}^+ s_{t,f}, s_{t,f} \rangle dt df = \int_\tau \int_\nu |S_{\mathbf{H}^+}(\tau, \nu)|^2 A_s^{(\alpha)*}(\tau, \nu) d\tau d\nu;$$

furthermore, with $\|s_{t,f}\|_2^2 = \|s\|_2^2 = 1$ we have $\int_t \int_f |L_{\mathbf{H}^+}^{(\alpha)}(t, f)|^2 \|s_{t,f}\|_2^2 dt df = \|L_{\mathbf{H}^+}^{(\alpha)}\|_2^2 = \|\mathbf{S}_{\mathbf{H}}\|_2^2$. Inserting these three results into (33) yields

$$\begin{aligned} \|\Delta_A^{(\alpha)}\|_2^2 &\leq \|x\|_2^2 \left[2 \int_\tau \int_\nu |S_{\mathbf{H}}(\tau, \nu)|^2 d\tau d\nu - 2 \text{Re}\left\{ \int_\tau \int_\nu |S_{\mathbf{H}}(\tau, \nu)|^2 A_s^{(\alpha)*}(\tau, \nu) d\tau d\nu \right\} \right] \\ &= 2 \|x\|_2^2 \int_\tau \int_\nu |S_{\mathbf{H}}(\tau, \nu)|^2 \left[1 - \text{Re}\{A_s^{(\alpha)}(\tau, \nu)\} \right] d\tau d\nu = 2 \|x\|_2^2 \|\mathbf{H}\|_2^2 [M_{\mathbf{H}}^{(\phi'_s)}]^2. \end{aligned} \quad (34)$$

The L_2 norm of $\Delta_B^{(\alpha)}(t, f)$ can be bounded using (13),

$$\|\Delta_B^{(\alpha)}\|_2^2 \leq \int_t \int_f |L_{\mathbf{H}^+}^{(\alpha)}(t, f) - L_{\mathbf{H}}^{(\alpha)*}(t, f)|^2 \|x\|_2^2 \|s_{t,f}\|_2^2 dt df \leq \|x\|_2^2 \|\mathbf{H}\|_2^2 [4\pi|\alpha|M_{\mathbf{H}}^{(1,1)}]^2. \quad (35)$$

The second (L_2) bound in (29) then follows by inserting (34) and (35) into $\|\Delta_\delta^{(\alpha)}\|_2 \leq \|\Delta_A^{(\alpha)}\|_2 + \|\Delta_B^{(\alpha)}\|_2$ (cf. (30)). \square

Thus, if $D_{\mathbf{H}^+,s}^{(\alpha)}$ (see Subsection III D) and $M_{\mathbf{H}}^{(\phi'_s)}$ can be made small by suitable choice of $s(t)$, and if $m_{\mathbf{H}}^{(1,1)}$ and $M_{\mathbf{H}}^{(1,1)}$ are small, we obtain the approximate input-output relation

$$\text{STFT}_{\mathbf{H}x}^{(s)}(t, f) \approx L_{\mathbf{H}}^{(\alpha)}(t, f) \text{STFT}_x^{(s)}(t, f).$$

In particular, small $M_{\mathbf{H}}^{(\phi'_s)}$ requires that $\text{Re}\{A_s^{(\alpha)}(\tau, \nu)\} \approx A_s^{(\alpha)}(0, 0) \equiv 1$ on the effective support of $|S_{\mathbf{H}}(\tau, \nu)|$, thus implying that this effective support is small, i.e., that \mathbf{H} is underspread. Note that the bounds in (29) simplify, and are tightest, in the case $\alpha = 0$.

F. Performance of the Short-Time Fourier Transform Filter

The relation $\text{STFT}_{\mathbf{H}x}^{(s)}(t, f) \approx L_{\mathbf{H}}^{(\alpha)}(t, f) \text{STFT}_x^{(s)}(t, f)$ valid in the underspread case suggests that the output signal $(\mathbf{H}x)(t)$ can approximately be obtained by applying an inverse short-time Fourier transform to $L_{\mathbf{H}}^{(\alpha)}(t, f) \text{STFT}_x^{(s)}(t, f)$. This corresponds to an approximate time-frequency implementation of \mathbf{H} that consists of calculating the short-time Fourier transform of the input signal, $\text{STFT}_x^{(s)}(t, f)$, multiplying by the time-frequency transfer function $L_{\mathbf{H}}^{(\alpha)}(t, f)$, and applying the inverse short-time Fourier transform (27) to $L_{\mathbf{H}}^{(\alpha)}(t, f) \text{STFT}_x^{(s)}(t, f)$. The resulting time-varying system will be called a *short-time Fourier transform filter* and denoted by $\widehat{\mathbf{H}}^{(\alpha)}$. Using (27) and (28), the input-output relation of the short-time Fourier transform filter is

$$(\widehat{\mathbf{H}}^{(\alpha)}x)(t) = \int_{t'} \int_{f'} \langle x, s_{t',f'} \rangle L_{\mathbf{H}}^{(\alpha)}(t', f') s_{t',f'}(t) dt' df'.$$

Such short-time Fourier transform filters (possibly with weighting functions other than $L_{\mathbf{H}}^{(\alpha)}(t, f)$) have been considered previously^{17,22–24,42,45,46,105,106}. They are an intuitively appealing, practical way of implementing a time-varying filter. In a time-frequency-discretized form, they permit time-varying subband signal processing using the Gabor expansion^{12,19,107,108} or, equivalently, DFT filter banks^{23,109–114}. Furthermore, in a mathematical context short-time Fourier transform filters are a special case of Toeplitz operators^{42,98,115}.

In the following, we first establish a relation and bound for the difference between the generalized Weyl symbol of the original operator \mathbf{H} and that of the short-time Fourier transform filter $\widehat{\mathbf{H}}^{(\alpha)}$; subsequently, we will bound the difference between the exact output signal $(\mathbf{H}x)(t)$ and the output signal $(\widehat{\mathbf{H}}^{(\alpha)}x)(t)$ of the short-time Fourier transform filter. We note that the L_∞ bound in the next theorem is an extension and adaptation of an existing bound^{42,44}.

Theorem III.8. *For any operator \mathbf{H} , the difference $\Delta_\tau^{(\alpha)}(t, f) \triangleq L_{\mathbf{H}}^{(\alpha)}(t, f) - L_{\widehat{\mathbf{H}}^{(\alpha)}}^{(\alpha)}(t, f)$ satisfies*

$$\frac{|\Delta_\tau^{(\alpha)}(t, f)|}{\|S_{\mathbf{H}}\|_1} \leq m_{\mathbf{H}}^{(\phi_s)}, \quad \frac{\|\Delta_\tau^{(\alpha)}\|_2}{\|\mathbf{H}\|_2} = M_{\mathbf{H}}^{(\phi_s)},$$

with the weighting function $\phi_s(\tau, \nu) = |1 - A_s^{(\alpha)}(\tau, \nu)|$ where $s(t)$ is the normalized window used in the short-time Fourier transform.

Proof. The spreading function of $\widehat{\mathbf{H}}^{(\alpha)}$ can be written as^{15,42,46,64} $S_{\widehat{\mathbf{H}}^{(\alpha)}}^{(\alpha)}(\tau, \nu) = S_{\mathbf{H}}^{(\alpha)}(\tau, \nu) A_s^{(\alpha)*}(\tau, \nu)$. The first (L_∞) bound is then shown as

$$\begin{aligned} |\Delta_\tau^{(\alpha)}(t, f)| &= \left| \int_\tau \int_\nu \left[S_{\mathbf{H}}^{(\alpha)}(\tau, \nu) - S_{\widehat{\mathbf{H}}^{(\alpha)}}^{(\alpha)}(\tau, \nu) \right] e^{i2\pi(\nu t - \tau f)} d\tau d\nu \right| \\ &\leq \int_\tau \int_\nu |S_{\mathbf{H}}(\tau, \nu)| \left| 1 - A_s^{(\alpha)*}(\tau, \nu) \right| d\tau d\nu = \|S_{\mathbf{H}}\|_1 m_{\mathbf{H}}^{(\phi_s)}. \end{aligned}$$

The expression for the L_2 norm of $\Delta_7^{(\alpha)}(t, f)$ is shown by noting that

$$\|\Delta_7^{(\alpha)}\|_2^2 = \left\| S_{\mathbf{H}}^{(\alpha)} - S_{\widehat{\mathbf{H}}^{(\alpha)}}^{(\alpha)} \right\|_2^2 = \int_{\tau} \int_{\nu} \left| S_{\mathbf{H}}^{(\alpha)}(\tau, \nu) \left[1 - A_s^{(\alpha)*}(\tau, \nu) \right] \right|^2 d\tau d\nu$$

and proceeding similarly. \square

Hence, if $m_{\mathbf{H}}^{(\phi_s)}$ and $M_{\mathbf{H}}^{(\phi_s)}$ can be made small by suitable choice of $s(t)$, then $L_{\mathbf{H}}^{(\alpha)}(t, f) \approx L_{\widehat{\mathbf{H}}^{(\alpha)}}^{(\alpha)}(t, f)$. Furthermore, since $\|\Delta_7^{(\alpha)}\|_2$ equals the Hilbert-Schmidt norm $\|\mathbf{H} - \widehat{\mathbf{H}}^{(\alpha)}\|_2$ and since the operator norm⁵⁹ $\|\mathbf{H} - \widehat{\mathbf{H}}^{(\alpha)}\|$ is bounded from above by the Hilbert-Schmidt norm, we obtain

$$\|\mathbf{H} - \widehat{\mathbf{H}}^{(\alpha)}\| \leq \|\mathbf{H} - \widehat{\mathbf{H}}^{(\alpha)}\|_2 = \|\mathbf{H}\|_2 M_{\mathbf{H}}^{(\phi_s)}. \quad (36)$$

Therefore, if $M_{\mathbf{H}}^{(\phi_s)}$ is small the short-time Fourier transform filter $\widehat{\mathbf{H}}^{(\alpha)}$ will be close to \mathbf{H} , i.e., $\widehat{\mathbf{H}}^{(\alpha)} \approx \mathbf{H}$. This is also shown by the following bounds on the difference of the respective output signals.

Corollary III.9. *For any operator \mathbf{H} and any signal $x(t)$, the difference $\Delta_8^{(\alpha)}(t) \triangleq (\mathbf{H}x)(t) - (\widehat{\mathbf{H}}^{(\alpha)}x)(t)$ is bounded as⁶*

$$\frac{|\Delta_8^{(\alpha)}(t)|}{\|S_{\mathbf{H}}\|_1 \|x\|_{\infty}} \leq m_{\mathbf{H}}^{(\phi_s)}, \quad \frac{\|\Delta_8^{(\alpha)}\|_2}{\|\mathbf{H}\|_2 \|x\|_2} \leq M_{\mathbf{H}}^{(\phi_s)},$$

with the weighting function $\phi_s(\tau, \nu)$ as in Theorem III.8.

Proof. Using (5), the relation $S_{\widehat{\mathbf{H}}^{(\alpha)}}^{(\alpha)}(\tau, \nu) = S_{\mathbf{H}}^{(\alpha)}(\tau, \nu) A_s^{(\alpha)*}(\tau, \nu)$, and $|x_{\tau, \nu}^{(\alpha)}(t)| \leq \|x_{\tau, \nu}^{(\alpha)}\|_{\infty} = \|x\|_{\infty}$, the first (L_{∞}) bound is shown as

$$\begin{aligned} |\Delta_8^{(\alpha)}(t)| &= \left| \int_{\tau} \int_{\nu} \left[S_{\mathbf{H}}^{(\alpha)}(\tau, \nu) - S_{\widehat{\mathbf{H}}^{(\alpha)}}^{(\alpha)}(\tau, \nu) \right] x_{\tau, \nu}^{(\alpha)}(t) d\tau d\nu \right| \\ &\leq \int_{\tau} \int_{\nu} |S_{\mathbf{H}}(\tau, \nu)| \left| 1 - A_s^{(\alpha)*}(\tau, \nu) \right| |x_{\tau, \nu}^{(\alpha)}(t)| d\tau d\nu \leq \|x\|_{\infty} \|S_{\mathbf{H}}\|_1 m_{\mathbf{H}}^{(\phi_s)}. \end{aligned}$$

The second (L_2) bound is shown by noting that

$$\|\Delta_8^{(\alpha)}\|_2^2 = \|(\mathbf{H} - \widehat{\mathbf{H}}^{(\alpha)})x\|_2^2 \leq \|\mathbf{H} - \widehat{\mathbf{H}}^{(\alpha)}\|^2 \|x\|_2^2$$

and using (36). \square

Thus, for an underspread operator \mathbf{H} for which $m_{\mathbf{H}}^{(\phi_s)}$ and $M_{\mathbf{H}}^{(\phi_s)}$ can be made small by suitable choice of the short-time Fourier transform window $s(t)$, we have $(\mathbf{H}x)(t) \approx (\widehat{\mathbf{H}}^{(\alpha)}x)(t)$, i.e., the output of the short-time Fourier transform filter is a good approximation to the output of the given operator \mathbf{H} .

G. Input-Output Relation for Random Signals

When a wide-sense stationary random process $x(t)$ with power spectral density¹¹⁶ $P_x(f)$ is passed through a time-invariant system \mathbf{H} with frequency response $G(f)$, the output $(\mathbf{H}x)(t)$ is again wide-sense stationary with power spectral density $P_{\mathbf{H}x}(f) = |G(f)|^2 P_x(f)$. Similarly, the response of a frequency-invariant system with temporal transfer function $m(t)$ to a (generally nonstationary) white process $x(t)$ with mean instantaneous intensity¹¹⁶ $q_x(t)$ is again white, with mean instantaneous intensity $q_{\mathbf{H}x}(t) = |m(t)|^2 q_x(t)$. We shall now investigate whether the simple multiplicative “input-output relations” $P_{\mathbf{H}x}(f) = |G(f)|^2 P_x(f)$ and $q_{\mathbf{H}x}(t) = |m(t)|^2 q_x(t)$, which relate the second-order statistics of $x(t)$ with those of $(\mathbf{H}x)(t)$, can be extended to the general time-varying/nonstationary case.

⁶Here, $\|x\|_{\infty} = \sup_t |x(t)|$ denotes the L_{∞} (sup) norm of $x(t)$.

Let us consider an operator \mathbf{H} whose input $x(t)$ is a general zero-mean, nonstationary random process. The *correlation operator* of $x(t)$ is defined as $\mathbf{R}_x \triangleq \mathbb{E}\{x \otimes x^*\}$, where \mathbb{E} denotes the expectation operator and $x \otimes x^*$ is the rank-one operator whose kernel is $x(t)x^*(t')$. Hence, the kernel of \mathbf{R}_x is the correlation function $r_x(t, t') = \mathbb{E}\{x(t)x^*(t')\}$. The system output $(\mathbf{H}x)(t)$ is a zero-mean, nonstationary random process whose correlation operator is given by

$$\mathbf{R}_{\mathbf{H}x} = \mathbb{E}\{(\mathbf{H}x) \otimes (\mathbf{H}x)^*\} = \mathbf{H} \mathbb{E}\{x \otimes x^*\} \mathbf{H}^\dagger = \mathbf{H} \mathbf{R}_x \mathbf{H}^\dagger.$$

We now look for a time-frequency reformulation of the above input-output relation in terms of the generalized Weyl symbol. Specifically, if \mathbf{H} and \mathbf{R}_x are *jointly underspread* in the sense of Subsection III C, then (19) and (14) imply (similar results have been obtained previously^{42,54})

$$L_{\mathbf{R}_{\mathbf{H}x}}^{(\alpha)}(t, f) = L_{\mathbf{H} \mathbf{R}_x \mathbf{H}^\dagger}^{(\alpha)}(t, f) \approx L_{\mathbf{H}}^{(\alpha)}(t, f) L_{\mathbf{R}_x}^{(\alpha)}(t, f) L_{\mathbf{H}^\dagger}^{(\alpha)}(t, f) \approx |L_{\mathbf{H}}^{(\alpha)}(t, f)|^2 L_{\mathbf{R}_x}^{(\alpha)}(t, f),$$

or equivalently

$$\overline{W}_{\mathbf{H}x}^{(\alpha)}(t, f) \approx |L_{\mathbf{H}}^{(\alpha)}(t, f)|^2 \overline{W}_x^{(\alpha)}(t, f), \quad (37)$$

where $\overline{W}_x^{(\alpha)}(t, f)$ denotes the generalized Wigner-Ville spectrum^{86,90,117} of a nonstationary process $x(t)$. The generalized Wigner-Ville spectrum is defined as the generalized Weyl symbol of the correlation operator,

$$\overline{W}_x^{(\alpha)}(t, f) \triangleq L_{\mathbf{R}_x}^{(\alpha)}(t, f) = \int_{\tau} r_x^{(\alpha)}(t, \tau) e^{-i2\pi f \tau} d\tau,$$

with $r_x^{(\alpha)}(t, \tau)$ obtained from $r_x(t, t')$ according to (3). It is a second-order time-frequency representation (“time-dependent power spectrum”) of nonstationary random processes. It reduces to the power spectral density $P_x(f)$ in the case of a wide-sense stationary process and to the mean instantaneous intensity $q_x(t)$ in the case of a (nonstationary) white process. The next theorem bounds the error of the approximation (37).

Theorem III.10. *For any operator \mathbf{H} and any (generally nonstationary and nonwhite) random process $x(t)$ with correlation operator \mathbf{R}_x , the difference $\Delta_9^{(\alpha)}(t, f) \triangleq \overline{W}_{\mathbf{H}x}^{(\alpha)}(t, f) - |L_{\mathbf{H}}^{(\alpha)}(t, f)|^2 \overline{W}_x^{(\alpha)}(t, f)$ is bounded as*

$$\frac{|\Delta_9^{(\alpha)}(t, f)|}{\|S_{\mathbf{H}}\|_1^2 \|S_{\mathbf{R}_x}\|_1} \leq 2\pi c_\alpha \left[m_{\mathbf{H}}^{(0,1)} m_{\mathbf{R}_x}^{(1,0)} + m_{\mathbf{H}}^{(1,0)} m_{\mathbf{R}_x}^{(0,1)} + m_{\mathbf{H}}^{(0,1)} m_{\mathbf{H}}^{(1,0)} \right] + 4\pi |\alpha| m_{\mathbf{H}}^{(1,1)} \quad (38)$$

with $c_\alpha = |\alpha + 1/2| + |\alpha - 1/2|$.

Proof. Subtracting and adding $L_{\mathbf{H}}^{(\alpha)}(t, f) L_{\mathbf{R}_x}^{(\alpha)}(t, f) L_{\mathbf{H}^\dagger}^{(\alpha)}(t, f)$, we have

$$|\Delta_9^{(\alpha)}(t, f)| = |\Delta_A^{(\alpha)}(t, f) + \Delta_B^{(\alpha)}(t, f)| \leq |\Delta_A^{(\alpha)}(t, f)| + |\Delta_B^{(\alpha)}(t, f)| \quad (39)$$

with

$$\Delta_A^{(\alpha)}(t, f) \triangleq L_{\mathbf{H} \mathbf{R}_x \mathbf{H}^\dagger}^{(\alpha)}(t, f) - L_{\mathbf{H}}^{(\alpha)}(t, f) L_{\mathbf{R}_x}^{(\alpha)}(t, f) L_{\mathbf{H}^\dagger}^{(\alpha)}(t, f) \quad (40)$$

$$\Delta_B^{(\alpha)}(t, f) \triangleq L_{\mathbf{H}}^{(\alpha)}(t, f) L_{\mathbf{R}_x}^{(\alpha)}(t, f) \left[L_{\mathbf{H}^\dagger}^{(\alpha)}(t, f) - L_{\mathbf{H}}^{(\alpha)*}(t, f) \right]. \quad (41)$$

Since according to (18) $S_{\mathbf{H} \mathbf{R}_x \mathbf{H}^\dagger}^{(\alpha)}(\tau, \nu) = (S_{\mathbf{H}}^{(\alpha)} \natural S_{\mathbf{R}_x}^{(\alpha)} \natural S_{\mathbf{H}^\dagger}^{(\alpha)})(\tau, \nu)$ and since the 2-D Fourier transform of $L_{\mathbf{H}}^{(\alpha)}(t, f) L_{\mathbf{R}_x}^{(\alpha)}(t, f) L_{\mathbf{H}^\dagger}^{(\alpha)}(t, f)$ is given by $(S_{\mathbf{H}}^{(\alpha)} * * S_{\mathbf{R}_x}^{(\alpha)} * * S_{\mathbf{H}^\dagger}^{(\alpha)})(\tau, \nu)$, the 2-D Fourier transform of $\Delta_A^{(\alpha)}(t, f)$ in (40) is obtained as

$$\begin{aligned} \hat{\Delta}_A^{(\alpha)}(\tau, \nu) &= (S_{\mathbf{H}}^{(\alpha)} \natural S_{\mathbf{R}_x}^{(\alpha)} \natural S_{\mathbf{H}^\dagger}^{(\alpha)})(\tau, \nu) - (S_{\mathbf{H}}^{(\alpha)} * * S_{\mathbf{R}_x}^{(\alpha)} * * S_{\mathbf{H}^\dagger}^{(\alpha)})(\tau, \nu) \\ &= \int_{\tau_1} \int_{\nu_1} \int_{\tau_2} \int_{\nu_2} S_{\mathbf{H}}^{(\alpha)}(\tau_1, \nu_1) S_{\mathbf{R}_x}^{(\alpha)}(\tau_2, \nu_2) S_{\mathbf{H}^\dagger}^{(\alpha)}(\tau - \tau_1 - \tau_2, \nu - \nu_1 - \nu_2) \\ &\quad \cdot \left[e^{-i2\pi[\phi_\alpha(\tau - \tau_1, \nu - \nu_1, \tau_2, \nu_2) + \phi_\alpha(\tau, \nu, \tau_1, \nu_1)]} - 1 \right] d\tau_1 d\nu_1 d\tau_2 d\nu_2. \end{aligned}$$

Using $|\sin x| \leq |x|$ and substituting $\tau_3 = \tau - \tau_1 - \tau_2$, $\nu_3 = \nu - \nu_1 - \nu_2$, the first term in (39) can then be bounded as

$$\begin{aligned}
|\Delta_A^{(\alpha)}(t, f)| &\leq \int_{\tau} \int_{\nu} |\hat{\Delta}_A^{(\alpha)}(\tau, \nu)| d\tau d\nu \\
&\leq 2\pi \int_{\tau_1} \int_{\nu_1} \int_{\tau_2} \int_{\nu_2} \int_{\tau_3} \int_{\nu_3} |S_{\mathbf{H}}(\tau_1, \nu_1)| |S_{\mathbf{R}_x}(\tau_2, \nu_2)| |S_{\mathbf{H}^+}(\tau_3, \nu_3)| \\
&\quad \cdot \left[\left| \alpha + \frac{1}{2} \right| [|\tau_1 \nu_2| + |\tau_1 \nu_3| + |\tau_2 \nu_3|] + \left| \alpha - \frac{1}{2} \right| [|\tau_2 \nu_1| + |\tau_3 \nu_1| + |\tau_3 \nu_2|] \right] \\
&\quad \cdot d\tau_1 d\nu_1 d\tau_2 d\nu_2 d\tau_3 d\nu_3 \\
&= 2\pi \|S_{\mathbf{H}}\|_1^2 \|S_{\mathbf{R}_x}\|_1 c_{\alpha} \left[m_{\mathbf{H}}^{(0,1)} m_{\mathbf{R}_x}^{(1,0)} + m_{\mathbf{H}}^{(1,0)} m_{\mathbf{R}_x}^{(0,1)} + m_{\mathbf{H}}^{(0,1)} m_{\mathbf{H}}^{(1,0)} \right], \tag{42}
\end{aligned}$$

where the final expression is obtained by collecting corresponding terms in the integral and using $m_{\mathbf{H}^+}^{(k,l)} = m_{\mathbf{H}}^{(k,l)}$. A bound on the second term in (39) is obtained by using $|L_{\mathbf{H}}^{(\alpha)}(t, f)| \leq \|S_{\mathbf{H}}\|_1$ in (41) and applying (13),

$$|\Delta_B^{(\alpha)}(t, f)| \leq \|S_{\mathbf{H}}\|_1 \|S_{\mathbf{R}_x}\|_1 \left| L_{\mathbf{H}^+}^{(\alpha)}(t, f) - L_{\mathbf{H}}^{(\alpha)*}(t, f) \right| \leq \|S_{\mathbf{H}}\|_1^2 \|S_{\mathbf{R}_x}\|_1 4\pi |\alpha| m_{\mathbf{H}}^{(1,1)}. \tag{43}$$

Inserting (42) and (43) in (39) yields the bound (38). \square

The theorem shows that the approximate input-output relation (37) will be valid if \mathbf{H} and \mathbf{R}_x are jointly underspread such that the moments appearing in the bound (38) are small. This requires the spreading functions of \mathbf{H} and \mathbf{R}_x to be similarly concentrated along the τ or ν axis. The bound (38) is tightest for $\alpha = 0$, in which case also a refined bound similar to (20) and (23) can be obtained by using metaplectic transformations of \mathbf{H} and \mathbf{R}_x ; this allows the spreading functions of \mathbf{H} and \mathbf{R}_x to be oriented in (similar) oblique directions.

H. Infimum and Supremum of the Weyl Symbol

We have seen in Subsection III D that the generalized Weyl symbol of an underspread operator can be interpreted as an approximate eigenvalue distribution over the time-frequency plane. We now assume \mathbf{H} to be self-adjoint, such that the eigenvalues $\lambda_{\mathbf{H},k}$ of \mathbf{H} are real-valued. Furthermore, we consider the Weyl symbol $L_{\mathbf{H}}^{(0)}(t, f)$ of \mathbf{H} which is real-valued as well. We shall investigate how close $\inf_{t,f} L_{\mathbf{H}}^{(0)}(t, f)$ and $\sup_{t,f} L_{\mathbf{H}}^{(0)}(t, f)$ are to $\lambda_{\mathbf{H}}^{\inf} \triangleq \inf_k \lambda_{\mathbf{H},k}$ and $\lambda_{\mathbf{H}}^{\sup} \triangleq \sup_k \lambda_{\mathbf{H},k}$, respectively. This issue will be seen to be of importance in Subsections III I and IV B. Related issues in the theory of quantization and pseudo-differential operators are the boundedness or positivity of operators corresponding to bounded or positive symbols, the results being theorems of the Calderón-Vaillancourt type^{15,40,41,118} and Gårding inequalities^{14,15,40,97}, respectively. The next theorem is an extension and adaptation of previous results^{42,44}.

Theorem III.11. *For any self-adjoint operator \mathbf{H} , there is*

$$\frac{\left| \inf_{t,f} L_{\mathbf{H}}^{(0)}(t, f) - \lambda_{\mathbf{H}}^{\inf} \right|}{\|S_{\mathbf{H}}\|_1} \leq m_{\mathbf{H}}^{(\phi_s)}, \quad \frac{\left| \sup_{t,f} L_{\mathbf{H}}^{(0)}(t, f) - \lambda_{\mathbf{H}}^{\sup} \right|}{\|S_{\mathbf{H}}\|_1} \leq m_{\mathbf{H}}^{(\phi_s)}, \tag{44}$$

with the weighting function $\phi_s(\tau, \nu) = \left| 1 - \frac{1}{A_s^{(0)}(\tau, \nu)} \right|$ where $s(t)$ is an arbitrary normalized function.

Proof. Let us define^{15,42,44} the “lower symbol” $L_{\mathbf{H}}^L(t, f) \triangleq \int_{\tau} \int_{\nu} S_{\mathbf{H}}^{(0)}(\tau, \nu) A_s^{(0)*}(\tau, \nu) e^{i2\pi(t\nu - f\tau)} d\tau d\nu$ and the “upper symbol” $L_{\mathbf{H}}^U(t, f) \triangleq \int_{\tau} \int_{\nu} \frac{S_{\mathbf{H}}^{(0)}(\tau, \nu)}{A_s^{(0)}(\tau, \nu)} e^{i2\pi(t\nu - f\tau)} d\tau d\nu$, where $s(t)$ is an arbitrary normalized function. (For Gaussian $s(t)$, the lower and upper symbols can be related to the *Wick and anti-Wick symbols* of quantum mechanics^{15,119} which can be defined using the normal and anti-normal correspondence rules^{34,36,83}.) It is known^{15,42} that for self-adjoint operators \mathbf{H}

$$\inf_{t,f} L_{\mathbf{H}}^U(t, f) \leq \lambda_{\mathbf{H}}^{\inf} \leq \inf_{t,f} L_{\mathbf{H}}^L(t, f) \tag{45}$$

$$\sup_{t,f} L_{\mathbf{H}}^L(t, f) \leq \lambda_{\mathbf{H}}^{\sup} \leq \sup_{t,f} L_{\mathbf{H}}^U(t, f). \tag{46}$$

We now have

$$\begin{aligned} \left| L_{\mathbf{H}}^{(0)}(t, f) - L_{\mathbf{H}}^L(t, f) \right| &= \left| \int_{\tau} \int_{\nu} S_{\mathbf{H}}^{(0)}(\tau, \nu) \left[1 - A_s^{(0)*}(\tau, \nu) \right] e^{i2\pi(\tau f - \nu t)} d\tau d\nu \right| \\ &\leq \int_{\tau} \int_{\nu} |S_{\mathbf{H}}(\tau, \nu)| \left| 1 - A_s^{(0)}(\tau, \nu) \right| d\tau d\nu = \|S_{\mathbf{H}}\|_1 m_{\mathbf{H}}^{(\tilde{\phi}_s)}, \end{aligned} \quad (47)$$

with $\tilde{\phi}_s(\tau, \nu) = |1 - A_s^{(0)}(\tau, \nu)|$. Similarly, we can show

$$\left| L_{\mathbf{H}}^{(0)}(t, f) - L_{\mathbf{H}}^U(t, f) \right| \leq \|S_{\mathbf{H}}\|_1 m_{\mathbf{H}}^{(\phi_s)}, \quad (48)$$

with $\phi_s(\tau, \nu) = \left| 1 - \frac{1}{A_s^{(0)}(\tau, \nu)} \right|$. Furthermore, due to $|A_s^{(0)}(\tau, \nu)| \leq \|s\|_2^2 = 1$ there is

$$\begin{aligned} m_{\mathbf{H}}^{(\phi_s)} &= \frac{1}{\|S_{\mathbf{H}}\|_1} \int_{\tau} \int_{\nu} \frac{|1 - A_s^{(0)}(\tau, \nu)|}{|A_s^{(0)}(\tau, \nu)|} |S_{\mathbf{H}}(\tau, \nu)| d\tau d\nu \\ &\geq \frac{1}{\|S_{\mathbf{H}}\|_1} \int_{\tau} \int_{\nu} |1 - A_s^{(0)}(\tau, \nu)| |S_{\mathbf{H}}(\tau, \nu)| d\tau d\nu = m_{\mathbf{H}}^{(\tilde{\phi}_s)}. \end{aligned} \quad (49)$$

It can be shown that (47) implies $|\inf_{t,f} L_{\mathbf{H}}^{(0)}(t, f) - \inf_{t,f} L_{\mathbf{H}}^L(t, f)| \leq \|S_{\mathbf{H}}\|_1 m_{\mathbf{H}}^{(\tilde{\phi}_s)}$ and that (48) implies $|\inf_{t,f} L_{\mathbf{H}}^{(0)}(t, f) - \inf_{t,f} L_{\mathbf{H}}^U(t, f)| \leq \|S_{\mathbf{H}}\|_1 m_{\mathbf{H}}^{(\phi_s)}$. Combining these two inequalities with (45) and (49), it can be shown that $|\inf_{t,f} L_{\mathbf{H}}^{(0)}(t, f) - \lambda_{\mathbf{H}}^{\text{inf}}| \leq \|S_{\mathbf{H}}\|_1 m_{\mathbf{H}}^{(\phi_s)}$. The bound on $|\sup_{t,f} L_{\mathbf{H}}^{(0)}(t, f) - \lambda_{\mathbf{H}}^{\text{sup}}|$ follows similarly using (47), (48), (46), and (49). \square

Hence, if $m_{\mathbf{H}}^{(\phi_s)}$ can be made small by suitable choice of $s(t)$, we have

$$\inf_{t,f} L_{\mathbf{H}}^{(0)}(t, f) \approx \lambda_{\mathbf{H}}^{\text{inf}}, \quad \sup_{t,f} L_{\mathbf{H}}^{(0)}(t, f) \approx \lambda_{\mathbf{H}}^{\text{sup}}.$$

Small $m_{\mathbf{H}}^{(\phi_s)}$ requires $A_s(\tau, \nu) \approx A_s(0, 0) \equiv 1$ on the effective support of $|S_{\mathbf{H}}(\tau, \nu)|$, thus implying that this effective support is small, i.e., that \mathbf{H} is underspread. We note that $m_{\mathbf{H}}^{(\phi_s)}$ can be adapted to oblique orientation of $|S_{\mathbf{H}}(\tau, \nu)|$ by proper choice of the normalized function $s(t)$.

I. Operator Norm (Maximum System Gain)

We now return to a general (i.e., not necessarily self-adjoint) operator \mathbf{H} . The operator norm (maximum system gain) of \mathbf{H} is defined as

$$\|\mathbf{H}\| \triangleq \sup_{\|x\|_2=1} \|\mathbf{H}x\|_2.$$

For time-invariant and frequency-invariant operators, $\|\mathbf{H}\|$ equals the supremum of the magnitude of the transfer function. We now ask whether for a general operator $\|\mathbf{H}\|$ is similarly related to the magnitude of $L_{\mathbf{H}}^{(\alpha)}(t, f)$.

Theorem III.12. *For any operator \mathbf{H} , the difference between the supremum of $|L_{\mathbf{H}}^{(\alpha)}(t, f)|^2$ and the squared operator norm is bounded as*

$$\frac{\left| \sup_{t,f} |L_{\mathbf{H}}^{(\alpha)}(t, f)|^2 - \|\mathbf{H}\|^2 \right|}{\|S_{\mathbf{H}}\|_1^2} \leq 2\pi m_{\mathbf{H}}^{(0,1)} m_{\mathbf{H}}^{(1,0)} + 4\pi |\alpha| m_{\mathbf{H}}^{(1,1)} + m_{\mathbf{H}+\mathbf{H}}^{(\phi_s)}, \quad (50)$$

with the weighting function $\phi_s(\tau, \nu)$ as in Theorem III.11.

Proof. Subtracting and adding $\sup_{t,f} |L_{\mathbf{H}}^{(0)}(t, f)|^2$ and $\sup_{t,f} L_{\mathbf{H}+\mathbf{H}}^{(0)}(t, f)$ from/to $\sup_{t,f} |L_{\mathbf{H}}^{(\alpha)}(t, f)|^2 - \|\mathbf{H}\|^2$, and using⁵⁹ $\|\mathbf{H}\|^2 = \lambda_{\mathbf{H}+\mathbf{H}}^{\text{sup}}$, we obtain

$$\begin{aligned}
\left| \sup_{t,f} |L_{\mathbf{H}}^{(\alpha)}(t, f)|^2 - \|\mathbf{H}\|^2 \right| &\leq \left| \sup_{t,f} |L_{\mathbf{H}}^{(\alpha)}(t, f)|^2 - \sup_{t,f} |L_{\mathbf{H}}^{(0)}(t, f)|^2 \right| \\
&\quad + \left| \sup_{t,f} |L_{\mathbf{H}}^{(0)}(t, f)|^2 - \sup_{t,f} L_{\mathbf{H}+\mathbf{H}}^{(0)}(t, f) \right| + \left| \sup_{t,f} L_{\mathbf{H}+\mathbf{H}}^{(0)}(t, f) - \lambda_{\mathbf{H}+\mathbf{H}}^{\text{sup}} \right|. \tag{51}
\end{aligned}$$

We next derive a bound on the difference $\Delta_{10}^{(\alpha)}(t, f) \triangleq |L_{\mathbf{H}}^{(\alpha)}(t, f)|^2 - |L_{\mathbf{H}}^{(0)}(t, f)|^2$ by noting that the 2-D Fourier transform of $|L_{\mathbf{H}}^{(\alpha)}(t, f)|^2$ is given by $S_{\mathbf{H}}^{(\alpha)}(\tau, \nu) ** S_{\mathbf{H}}^{(\alpha)*}(-\tau, -\nu)$ and by using $S_{\mathbf{H}}^{(\alpha)}(\tau, \nu) = S_{\mathbf{H}}^{(0)}(\tau, \nu) e^{-i2\pi\alpha\tau\nu}$,

$$\begin{aligned}
|\Delta_{10}^{(\alpha)}(t, f)| &\leq \int_{\tau} \int_{\nu} |\hat{\Delta}_{10}^{(\alpha)}(\tau, \nu)| d\tau d\nu \\
&= \int_{\tau} \int_{\nu} \left| \int_{\tau'} \int_{\nu'} S_{\mathbf{H}}^{(0)}(\tau', \nu') S_{\mathbf{H}}^{(0)*}(\tau' - \tau, \nu' - \nu) \left[e^{-i2\pi\alpha[\tau'\nu' - (\tau' - \tau)(\nu' - \nu)]} - 1 \right] d\tau' d\nu' \right| d\tau d\nu \\
&\leq 2 \int_{\tau_1} \int_{\nu_1} \int_{\tau'} \int_{\nu'} |S_{\mathbf{H}}(\tau', \nu')| |S_{\mathbf{H}}(\tau_1, \nu_1)| |\sin \pi\alpha(\tau'\nu' - \tau_1\nu_1)| d\tau' d\nu' d\tau_1 d\nu_1 \\
&\leq 2\pi|\alpha| \int_{\tau_1} \int_{\nu_1} \int_{\tau'} \int_{\nu'} |S_{\mathbf{H}}(\tau', \nu')| |S_{\mathbf{H}}(\tau_1, \nu_1)| [|\tau'\nu'| + |\tau_1\nu_1|] d\tau' d\nu' d\tau_1 d\nu_1 \\
&= 4\pi|\alpha| \|S_{\mathbf{H}}\|_1^2 m_{\mathbf{H}}^{(1,1)}.
\end{aligned}$$

It can be shown that this bound also implies that the first term in (51) is bounded as $\left| \sup_{t,f} |L_{\mathbf{H}}^{(\alpha)}(t, f)|^2 - \sup_{t,f} |L_{\mathbf{H}}^{(0)}(t, f)|^2 \right| \leq 4\pi|\alpha| \|S_{\mathbf{H}}\|_1^2 m_{\mathbf{H}}^{(1,1)}$. Furthermore, it follows from (22) with $\alpha = 0$ that the second term in (51) is bounded as $\left| \sup_{t,f} |L_{\mathbf{H}}^{(0)}(t, f)|^2 - \sup_{t,f} L_{\mathbf{H}+\mathbf{H}}^{(0)}(t, f) \right| \leq 2\pi \|S_{\mathbf{H}}\|_1^2 m_{\mathbf{H}}^{(0,1)} m_{\mathbf{H}}^{(1,0)}$. Finally, since $\mathbf{H}+\mathbf{H}$ is self-adjoint, the bounds in (44) hold for $\mathbf{H}+\mathbf{H}$ and we obtain that the third term in (51) is bounded as $\left| \sup_{t,f} L_{\mathbf{H}+\mathbf{H}}^{(0)}(t, f) - \lambda_{\mathbf{H}+\mathbf{H}}^{\text{sup}} \right| \leq \|S_{\mathbf{H}+\mathbf{H}}\|_1 m_{\mathbf{H}+\mathbf{H}}^{(\phi_s)} \leq \|S_{\mathbf{H}}\|_1^2 m_{\mathbf{H}+\mathbf{H}}^{(\phi_s)}$. Inserting these three bounds into (51) yields (50). \square

Hence, if $m_{\mathbf{H}+\mathbf{H}}^{(\phi_s)}$ can be made small by suitable choice of the function $s(t)$, and if $m_{\mathbf{H}}^{(1,1)}$ and $m_{\mathbf{H}}^{(0,1)} m_{\mathbf{H}}^{(1,0)}$ are small too, then

$$\sup_{t,f} |L_{\mathbf{H}}^{(\alpha)}(t, f)| \approx \|\mathbf{H}\|.$$

In particular, small $m_{\mathbf{H}+\mathbf{H}}^{(\phi_s)}$ requires that $A_s(\tau, \nu) \approx A_s(0, 0) \equiv 1$ on the effective support of $|S_{\mathbf{H}+\mathbf{H}}(\tau, \nu)|$, thus implying that $\mathbf{H}+\mathbf{H}$ is underspread, which will be true if \mathbf{H} is underspread.

The bound in (50) is tightest for $\alpha = 0$ since here the bound's second term vanishes. Moreover, using (23) we obtain the tighter bound

$$\frac{\left| \sup_{t,f} |L_{\mathbf{H}}^{(0)}(t, f)|^2 - \|\mathbf{H}\|^2 \right|}{\|S_{\mathbf{H}}\|_1^2} \leq 2\pi \min_{\mathbf{U} \in \mathcal{M}} \left\{ m_{\mathbf{U}\mathbf{H}\mathbf{U}}^{(0,1)} + m_{\mathbf{U}\mathbf{H}\mathbf{U}}^{(1,0)} \right\} + m_{\mathbf{H}+\mathbf{H}}^{(\phi_s)}. \tag{52}$$

The first term in this refined bound may be small even if the spreading function is oriented in an oblique direction in the (τ, ν) plane. The second term, $m_{\mathbf{H}+\mathbf{H}}^{(\phi_s)}$, can be adapted to oblique directions by proper choice of $s(t)$.

IV. PROPERTIES OF UNDERSPREAD OPERATORS

In this section, we show that besides permitting a simple approximate transfer function (symbolic) calculus, underspread operators have other “desirable” properties that may simplify their analysis.

A. Approximate Commutation

In contrast to the time-invariant or frequency-invariant case, two general linear operators \mathbf{G} and \mathbf{H} typically do not commute, i.e., $\mathbf{G}\mathbf{H} \neq \mathbf{H}\mathbf{G}$ or equivalently $[\mathbf{G}, \mathbf{H}] \triangleq \mathbf{G}\mathbf{H} - \mathbf{H}\mathbf{G} \neq \mathbf{0}$. The following theorem shows that this situation is somewhat different for jointly underspread operators.

Theorem IV.1. *The operator norm of the commutator of two operators \mathbf{G} and \mathbf{H} is bounded as*

$$\begin{aligned} \frac{\|[\mathbf{G}, \mathbf{H}]\|^2}{\|S_{\mathbf{G}}\|_1^2 \|S_{\mathbf{H}}\|_1^2} &\leq 16\pi^2 \left[\min_{\mathbf{U} \in \mathcal{M}} B_{\mathbf{U}\mathbf{G}\mathbf{U}^+, \mathbf{U}\mathbf{H}\mathbf{U}^+}^{(0)} \right]^2 \\ &\quad + 8\pi \min_{\mathbf{U} \in \mathcal{M}} \left\{ \left[m_{\mathbf{U}\mathbf{G}\mathbf{U}^+}^{(0,1)} + m_{\mathbf{U}\mathbf{H}\mathbf{U}^+}^{(0,1)} \right] \left[m_{\mathbf{U}\mathbf{G}\mathbf{U}^+}^{(1,0)} + m_{\mathbf{U}\mathbf{H}\mathbf{U}^+}^{(1,0)} \right] \right\} + 4m_{[\mathbf{G}, \mathbf{H}]^+[\mathbf{G}, \mathbf{H}]}^{(\phi_s)}, \end{aligned} \quad (53)$$

with $B_{\mathbf{G}, \mathbf{H}}^{(0)} = \frac{1}{2} [m_{\mathbf{G}}^{(0,1)} m_{\mathbf{H}}^{(1,0)} + m_{\mathbf{G}}^{(1,0)} m_{\mathbf{H}}^{(0,1)}]$ and with the weighting function $\phi_s(\tau, \nu)$ as in Theorem III.11.

Proof. Applying (52) to $[\mathbf{G}, \mathbf{H}]$ yields

$$\|[\mathbf{G}, \mathbf{H}]\|^2 \leq \sup_{t, f} |L_{[\mathbf{G}, \mathbf{H}]}^{(0)}(t, f)|^2 + \|S_{[\mathbf{G}, \mathbf{H}]}\|_1^2 \left[2\pi \min_{\mathbf{U} \in \mathcal{M}} \left\{ m_{\mathbf{U}[\mathbf{G}, \mathbf{H}]\mathbf{U}^+}^{(0,1)} + m_{\mathbf{U}[\mathbf{G}, \mathbf{H}]\mathbf{U}^+}^{(1,0)} \right\} + m_{[\mathbf{G}, \mathbf{H}]^+[\mathbf{G}, \mathbf{H}]}^{(\phi_s)} \right]. \quad (54)$$

In order to bound $\sup_{t, f} |L_{[\mathbf{G}, \mathbf{H}]}^{(0)}(t, f)|^2$, we consider $L_{[\mathbf{G}, \mathbf{H}]}^{(0)}(t, f)$. Subtracting and adding $L_{\mathbf{G}}^{(0)}(t, f)L_{\mathbf{H}}^{(0)}(t, f)$ and applying (20) twice, we obtain

$$\begin{aligned} |L_{[\mathbf{G}, \mathbf{H}]}^{(0)}(t, f)| &= |L_{\mathbf{G}\mathbf{H}}^{(0)}(t, f) - L_{\mathbf{H}\mathbf{G}}^{(0)}(t, f)| \\ &\leq |L_{\mathbf{G}\mathbf{H}}^{(0)}(t, f) - L_{\mathbf{G}}^{(0)}(t, f)L_{\mathbf{H}}^{(0)}(t, f)| + |L_{\mathbf{G}}^{(0)}(t, f)L_{\mathbf{H}}^{(0)}(t, f) - L_{\mathbf{H}\mathbf{G}}^{(0)}(t, f)| \\ &\leq 2\|S_{\mathbf{G}}\|_1 \|S_{\mathbf{H}}\|_1 2\pi \min_{\mathbf{U} \in \mathcal{M}} B_{\mathbf{U}\mathbf{G}\mathbf{U}^+, \mathbf{U}\mathbf{H}\mathbf{U}^+}^{(0)}, \end{aligned}$$

which gives the bound

$$\sup_{t, f} |L_{[\mathbf{G}, \mathbf{H}]}^{(0)}(t, f)|^2 \leq \|S_{\mathbf{G}}\|_1^2 \|S_{\mathbf{H}}\|_1^2 16\pi^2 \left[\min_{\mathbf{U} \in \mathcal{M}} B_{\mathbf{U}\mathbf{G}\mathbf{U}^+, \mathbf{U}\mathbf{H}\mathbf{U}^+}^{(0)} \right]^2. \quad (55)$$

With regard to the second term in (54), we have

$$\begin{aligned} \|S_{[\mathbf{G}, \mathbf{H}]}\|_1 m_{\mathbf{U}[\mathbf{G}, \mathbf{H}]\mathbf{U}^+}^{(0,1)} &= \|S_{\mathbf{U}[\mathbf{G}, \mathbf{H}]\mathbf{U}^+}\|_1 m_{\mathbf{U}[\mathbf{G}, \mathbf{H}]\mathbf{U}^+}^{(0,1)} \\ &\leq \int_{\tau} \int_{\nu} |\nu| [|S_{\mathbf{U}\mathbf{G}\mathbf{U}^+ \mathbf{U}\mathbf{H}\mathbf{U}^+}(\tau, \nu)| + |S_{\mathbf{U}\mathbf{H}\mathbf{U}^+ \mathbf{U}\mathbf{G}\mathbf{U}^+}(\tau, \nu)|] d\tau d\nu \\ &\leq \|S_{\mathbf{G}}\|_1 \|S_{\mathbf{H}}\|_1 2[m_{\mathbf{U}\mathbf{G}\mathbf{U}^+}^{(0,1)} + m_{\mathbf{U}\mathbf{H}\mathbf{U}^+}^{(0,1)}], \end{aligned}$$

where the last step can be shown using the twisted convolution (18). In a similar manner, one can show that $\|S_{[\mathbf{G}, \mathbf{H}]}\|_1 m_{\mathbf{U}[\mathbf{G}, \mathbf{H}]\mathbf{U}^+}^{(1,0)} \leq \|S_{\mathbf{G}}\|_1 \|S_{\mathbf{H}}\|_1 2[m_{\mathbf{U}\mathbf{G}\mathbf{U}^+}^{(1,0)} + m_{\mathbf{U}\mathbf{H}\mathbf{U}^+}^{(1,0)}]$. Inserting these bounds into (54) and using $\|S_{[\mathbf{G}, \mathbf{H}]}\|_1 \leq \|S_{\mathbf{G}\mathbf{H}}\|_1 + \|S_{\mathbf{H}\mathbf{G}}\|_1 \leq 2\|S_{\mathbf{G}}\|_1 \|S_{\mathbf{H}}\|_1$, the bound (53) follows. \square

Hence, the operator norm of the commutator of two jointly underspread operators is small, which shows that two jointly underspread operators approximately commute, i.e., $\mathbf{G}\mathbf{H} \approx \mathbf{H}\mathbf{G}$. Since two operators commute if and only if they have a common set of eigenfunctions^{15,59}, this result is consistent with Theorem III.6. While here we have considered the operator norm $\|[\mathbf{G}, \mathbf{H}]\|$, we note that bounds on the Hilbert-Schmidt norm $\|[\mathbf{G}, \mathbf{H}]\|_2$ have been formulated previously⁴².

Almost commuting operators may look peculiar to quantum physicists since in the classical limit ($\hbar \rightarrow 0$) of the quantum-mechanical symbolic calculus the Weyl symbol of the commutator $[\mathbf{G}, \mathbf{H}]$ reduces to the (generally non-zero) Poisson bracket of the symbols⁷ which is defined as¹⁵

$$\{L_{\mathbf{H}}^{(\alpha)}, L_{\mathbf{G}}^{(\alpha)}\} \triangleq \frac{\partial L_{\mathbf{H}}^{(\alpha)}(t, f)}{\partial t} \frac{\partial L_{\mathbf{G}}^{(\alpha)}(t, f)}{\partial f} - \frac{\partial L_{\mathbf{G}}^{(\alpha)}(t, f)}{\partial t} \frac{\partial L_{\mathbf{H}}^{(\alpha)}(t, f)}{\partial f}.$$

However, the following bound shows that the Poisson bracket of two jointly underspread operators is close to zero.

⁷Specifically, if the Weyl symbol is re-defined^{15,34,36,56} to include Planck's constant \hbar , then one can show^{15,56} that $L_{[\mathbf{G}, \mathbf{H}]}^{(0)} = -i\frac{\hbar}{2\pi} \{L_{\mathbf{G}}^{(0)}, L_{\mathbf{H}}^{(0)}\} + \mathcal{O}(\hbar^2)$, i.e., the Weyl symbol of $[\mathbf{G}, \mathbf{H}]$ is proportional to $\{L_{\mathbf{G}}^{(0)}, L_{\mathbf{H}}^{(0)}\}$ up to errors of second order.

Theorem IV.2. *The Poisson bracket of the generalized Weyl symbols of two operators \mathbf{G} and \mathbf{H} is bounded as*

$$\frac{|\{L_{\mathbf{G}}^{(\alpha)}, L_{\mathbf{H}}^{(\alpha)}\}|}{\|S_{\mathbf{G}}\|_1 \|S_{\mathbf{H}}\|_1} \leq 8\pi^2 \min_{\mathbf{U} \in \mathcal{M}} B_{\mathbf{UGU}^+, \mathbf{UHU}^+}^{(0)} \quad (56)$$

with $B_{\mathbf{G}, \mathbf{H}}^{(0)} = \frac{1}{2} [m_{\mathbf{G}}^{(0,1)} m_{\mathbf{H}}^{(1,0)} + m_{\mathbf{G}}^{(1,0)} m_{\mathbf{H}}^{(0,1)}]$.

Proof. Since differentiations and multiplications in the time-frequency domain correspond to multiplications and convolutions in the (τ, ν) domain, it is easily shown that the 2-D Fourier transform of the Poisson bracket is given by

$$\begin{aligned} \int_t \int_f \{L_{\mathbf{G}}^{(\alpha)}, L_{\mathbf{H}}^{(\alpha)}\} e^{-i2\pi(\nu t - \tau f)} dt df &= \int_{\tau'} \int_{\nu'} \left[i2\pi\nu' S_{\mathbf{G}}^{(\alpha)}(\tau', \nu') i2\pi(\tau - \tau') S_{\mathbf{H}}^{(\alpha)}(\tau - \tau', \nu - \nu') \right. \\ &\quad \left. - i2\pi\tau' S_{\mathbf{G}}^{(\alpha)}(\tau', \nu') i2\pi(\nu - \nu') S_{\mathbf{H}}^{(\alpha)}(\tau - \tau', \nu - \nu') \right] d\tau' d\nu'. \end{aligned}$$

Furthermore, the magnitude of the Poisson bracket is bounded by the L_1 norm of its Fourier transform, and therefore

$$\begin{aligned} |\{L_{\mathbf{G}}^{(\alpha)}, L_{\mathbf{H}}^{(\alpha)}\}| &\leq \int_{\tau} \int_{\nu} \left| 4\pi^2 \int_{\tau'} \int_{\nu'} [\nu'(\tau - \tau') - \tau'(\nu - \nu')] S_{\mathbf{G}}^{(\alpha)}(\tau', \nu') S_{\mathbf{H}}^{(\alpha)}(\tau - \tau', \nu - \nu') d\tau' d\nu' \right| d\tau d\nu \\ &\leq 4\pi^2 \int_{\tau_1} \int_{\nu_1} \int_{\tau_2} \int_{\nu_2} |\nu_1 \tau_2 - \tau_1 \nu_2| |S_{\mathbf{G}}(\tau_1, \nu_1)| |S_{\mathbf{H}}(\tau_2, \nu_2)| d\tau_1 d\nu_1 d\tau_2 d\nu_2 \\ &\leq 4\pi^2 \|S_{\mathbf{G}}\|_1 \|S_{\mathbf{H}}\|_1 \left[m_{\mathbf{G}}^{(0,1)} m_{\mathbf{H}}^{(1,0)} + m_{\mathbf{G}}^{(1,0)} m_{\mathbf{H}}^{(0,1)} \right]. \end{aligned} \quad (57)$$

Since $\nu_1 \tau_2 - \tau_1 \nu_2$ in (57) is invariant to symplectic coordinate transforms, the above bound remains valid if \mathbf{G} and \mathbf{H} are replaced by \mathbf{UGU}^+ and \mathbf{UHU}^+ , respectively, with $\mathbf{U} \in \mathcal{M}$. \square

Hence, if \mathbf{G} and \mathbf{H} are jointly underspread such that $\min_{\mathbf{U} \in \mathcal{M}} B_{\mathbf{UGU}^+, \mathbf{UHU}^+}^{(0)}$ is small, then the Poisson bracket $\{L_{\mathbf{G}}^{(\alpha)}, L_{\mathbf{H}}^{(\alpha)}\}$ is approximately zero, thereby confirming our result that the commutator of jointly underspread operators vanishes approximately. We note that the bound in (56) does not depend on α , and that it equals the bound in (20) up to a factor.

B. Approximate Normality

Normal operators have the advantage of allowing an eigenvalue decomposition instead of a (numerically more expensive) singular value decomposition. In contrast to time-invariant or frequency-invariant systems, general operators may be non-normal, i.e., $\mathbf{HH}^+ \neq \mathbf{H}^+\mathbf{H}$ or equivalently $[\mathbf{H}, \mathbf{H}^+] = \mathbf{HH}^+ - \mathbf{H}^+\mathbf{H} \neq \mathbf{0}$. A bound on $\|[\mathbf{H}, \mathbf{H}^+]\|$ could be obtained as a special case of (53). However, the following theorem exploits the self-adjointness of $[\mathbf{H}, \mathbf{H}^+]$ to yield a tighter and simpler bound.

Theorem IV.3. *The operator norm of the commutator of an operator \mathbf{H} and its adjoint \mathbf{H}^+ is bounded as*

$$\frac{\|[\mathbf{H}, \mathbf{H}^+]\|}{\|S_{\mathbf{H}}\|_1^2} \leq 4\pi \min_{\mathbf{U} \in \mathcal{M}} \left\{ m_{\mathbf{UHU}^+}^{(0,1)} + m_{\mathbf{UHU}^+}^{(1,0)} \right\} + m_{\mathbf{HH}^+}^{(\phi_s)} + m_{\mathbf{H}^+\mathbf{H}}^{(\phi_s)}, \quad (58)$$

with the weighting function $\phi_s(\tau, \nu)$ as in Theorem III.11.

Proof. It is known⁵⁹ that $\|[\mathbf{H}, \mathbf{H}^+]\| = \max\{-\lambda_{[\mathbf{H}, \mathbf{H}^+]}^{\inf}, \lambda_{[\mathbf{H}, \mathbf{H}^+]}^{\sup}\} = \max\{\lambda_{-[\mathbf{H}, \mathbf{H}^+]}^{\sup}, \lambda_{[\mathbf{H}, \mathbf{H}^+]}^{\sup}\}$. First assume that $\|[\mathbf{H}, \mathbf{H}^+]\| = \lambda_{[\mathbf{H}, \mathbf{H}^+]}^{\sup}$. In that case, the second bound in (44) yields

$$\begin{aligned} \|[\mathbf{H}, \mathbf{H}^+]\| &= \lambda_{[\mathbf{H}, \mathbf{H}^+]}^{\sup} \leq \sup_{t, f} L_{[\mathbf{H}, \mathbf{H}^+]}^{(0)}(t, f) + \|S_{[\mathbf{H}, \mathbf{H}^+]}\|_1 m_{[\mathbf{H}, \mathbf{H}^+]}^{(\phi_s)} \\ &\leq \sup_{t, f} \left| L_{[\mathbf{H}, \mathbf{H}^+]}^{(0)}(t, f) \right| + \|S_{[\mathbf{H}, \mathbf{H}^+]}\|_1 m_{[\mathbf{H}, \mathbf{H}^+]}^{(\phi_s)}. \end{aligned} \quad (59)$$

Specializing (55), the first term in (59) can be bounded as

$$\sup_{t,f} \left| L_{[\mathbf{H}, \mathbf{H}^+]}^{(0)}(t, f) \right| \leq \|S_{\mathbf{H}}\|_1^2 4\pi \min_{\mathbf{U} \in \mathcal{M}} m_{\mathbf{U}\mathbf{H}\mathbf{U}^+}^{(0,1)} m_{\mathbf{U}\mathbf{H}\mathbf{U}^+}^{(1,0)}, \quad (60)$$

where we used that $B_{\mathbf{H}, \mathbf{H}^+}^{(0)} = m_{\mathbf{H}}^{(0,1)} m_{\mathbf{H}}^{(1,0)}$ due to $m_{\mathbf{H}^+}^{(k,l)} = m_{\mathbf{H}}^{(k,l)}$. The second term in (59) can be bounded as

$$\begin{aligned} \|S_{[\mathbf{H}, \mathbf{H}^+]}\|_1 m_{[\mathbf{H}, \mathbf{H}^+]}^{(\phi_s)} &\leq \int_{\tau} \int_{\nu} \phi_s(\tau, \nu) [|S_{\mathbf{H}\mathbf{H}^+}(\tau, \nu)| + |S_{\mathbf{H}^+\mathbf{H}}(\tau, \nu)|] d\tau d\nu \\ &= \|S_{\mathbf{H}\mathbf{H}^+}\|_1 m_{\mathbf{H}\mathbf{H}^+}^{(\phi_s)} + \|S_{\mathbf{H}^+\mathbf{H}}\|_1 m_{\mathbf{H}^+\mathbf{H}}^{(\phi_s)} \leq \|S_{\mathbf{H}}\|_1^2 \left[m_{\mathbf{H}\mathbf{H}^+}^{(\phi_s)} + m_{\mathbf{H}^+\mathbf{H}}^{(\phi_s)} \right]. \end{aligned} \quad (61)$$

Inserting (60) and (61) into (59), the bound (58) follows. If $\|[\mathbf{H}, \mathbf{H}^+]\| = -\lambda_{[\mathbf{H}, \mathbf{H}^+]}^{\text{inf}} = \lambda_{-[\mathbf{H}, \mathbf{H}^+]}^{\text{sup}}$, the bound is shown similarly by applying the second bound in (44) to $-[\mathbf{H}, \mathbf{H}^+]$. \square

Hence, an underspread operator \mathbf{H} for which $\min_{\mathbf{U} \in \mathcal{M}} \{m_{\mathbf{U}\mathbf{H}\mathbf{U}^+}^{(0,1)} + m_{\mathbf{U}\mathbf{H}\mathbf{U}^+}^{(1,0)}\}$ as well as $m_{\mathbf{H}\mathbf{H}^+}^{(\phi_s)}$ and $m_{\mathbf{H}^+\mathbf{H}}^{(\phi_s)}$ are small satisfies $\mathbf{H}\mathbf{H}^+ \approx \mathbf{H}^+\mathbf{H}$ and is thus approximately normal. We note that bounds on the Hilbert-Schmidt norm $\|[\mathbf{H}, \mathbf{H}^+]\|_2$ have been formulated previously⁴².

V. CONCLUSION

We have introduced a new concept of “underspread” linear operators (linear time-varying systems) that does not require the support of the spreading function to be compact and thus extends a previous definition of underspread operators. While sufficiently general to be relevant to a broad variety of engineering applications, our underspread concept still is restrictive enough to permit the formulation of a simple and intuitive approximate transfer function calculus (symbolic calculus) that is based on the generalized Weyl symbol. Indeed, in the case of underspread operators the generalized Weyl symbol can be viewed as an approximate “time-frequency transfer function” that is similarly simple to use as the conventional transfer function of linear time-invariant systems. The quality and scope of validity of this transfer function approximation was assessed by providing explicit upper bounds on various approximation errors; these bounds are formulated in terms of weighted integrals and moments of the spreading function.

The transfer function calculus provides a foundation and justification of several methods which have been proposed to design and implement time-varying filters for signal enhancement, estimation, and detection^{45–53}. Furthermore, the calculus can immediately be applied to the theory of time-varying power spectra of nonstationary random processes. Here, the operators are either correlation operators or innovation systems (cf. Subsection III G). This application leads to the important concept of *underspread random processes* for which several important definitions of time-varying power spectra can be shown to become approximately equivalent^{42,54,55}.

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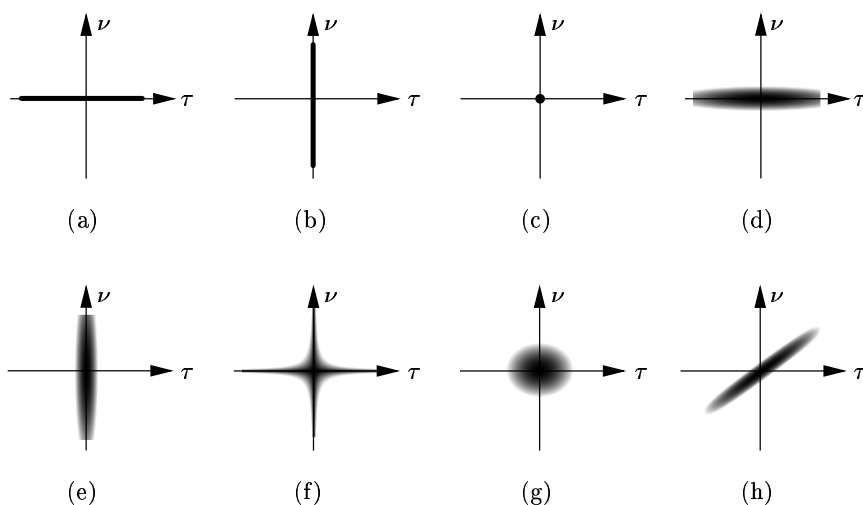


FIG. 1. Examples for the magnitude of the spreading function of some generic types of operators: (a) time-invariant system, (b) frequency-invariant system, (c) identity operator, (d) quasi-time-invariant system, (e) quasi-frequency-invariant system, (f)–(h) three different versions of underspread operators.

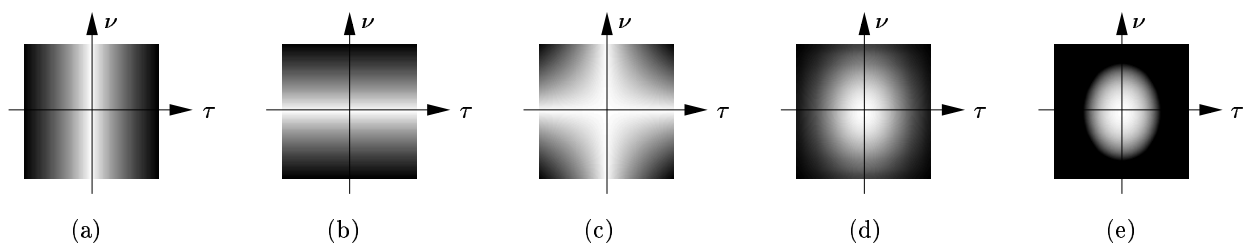


FIG. 2. Gray-scale plots (darker shades correspond to larger values) of some specific weighting functions: (a) $\phi(\tau, \nu) = |\tau|^k$, (b) $\phi(\tau, \nu) = |\nu|^k$, (c) $\phi(\tau, \nu) = |\tau\nu|^k$, (d) $\phi(\tau, \nu) = |1 - A_s^{(0)}(\tau, \nu)|$, (e) $\phi(\tau, \nu) = \left|1 - \frac{1}{A_s^{(0)}(\tau, \nu)}\right|$ with $A_s^{(0)}(\tau, \nu)$ the (symmetric) ambiguity function of a normalized Gaussian function (cf. Subsections III D and III H).

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