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# UNCERTAINTY AND CONCENTRATION INEQUALITIES FOR NONSTATIONARY RANDOM PROCESSES AND TIME-FREQUENCY ENERGY SPECTRA*

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## ABSTRACT

We provide various sharp uncertainty inequalities for nonstationary random processes. These inequalities relate the temporal and spectral energy concentration of a process to the "effective rank" of its correlation operator. Similar inequalities are established for the time-frequency (TF) concentration of a wide range of TF energy spectra. We further identify TF energy spectra that feature maximum TF concentration and illustrate our results for the practically important class of underspread random processes.

## 1. INTRODUCTION

There exist several uncertainty inequalities for deterministic signals, all of which express the fact that a signal cannot be arbitrarily concentrated in time and frequency simultaneously. *Heisenberg*-type inequalities [1–4] are based on using temporal and spectral moments as measures of the effective duration and effective bandwidth, respectively, while *Slepian–Pollak–Landau* (SPL) theory (cf. [4–7]) yields inequalities that are based on measuring the energy concentration of a signal within compact time and frequency intervals. It has also been shown that these uncertainty inequalities place corresponding lower bounds on the time-frequency (TF) resolution and TF concentration of joint TF energy distributions like the Wigner distribution, spectrogram etc. [2, 7–10].

In this paper, we establish uncertainty and concentration inequalities for nonstationary random processes and TF energy spectra. While such relations have not been available before, existing uncertainty inequalities for positive semi-definite operators [11–13] can be applied to the correlation operator $\mathbf{R}_x$ of a random processes $x(t)$ (the correlation operator is the linear operator whose kernel equals $r_x(t,t') = \mathbb{E}\{x(t)\,x^*(t')\}$). However, the resulting inequalities are weaker than the ones we present below. Our main contributions in this paper can be summarized as follows:

- we provide sharp Heisenberg-type uncertainty inequalities for nonstationary processes where the lower bounds depend on a measure of the effective rank of the correlation operator $\mathbf{R}_x$;
- we derive sharp uncertainty inequalities for nonstationary random processes using SPL theory; these inequalities relate the temporal/spectral concentration of a random process to the eigenvalue spread of its correlation operator;
- we establish lower bounds on the TF spread of *type I TF energy spectra* [7, 14–16] like the *Wigner-Ville spectrum* [7, 17], the *Rihaczek spectrum* [18], the *instantaneous power spectrum* [19], and the *physical spectrum* (i.e., expected spectrogram) [20];
- similar inequalities are presented for *type II TF energy spectra* [14, 16, 21] like the *Weyl spectrum* [21], the *evolutionary spectrum* [3, 21] *and the transitory evolutionary spectrum* [21, 22].

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Throughout the paper, we consider zero-mean nonstationary random processes $x(t)$ which are characterized by their correlation operator $\mathbf{R}_x$. We assume that $x(t)$ has finite mean energy $\bar{E}_x \triangleq \mathbb{E}\{\|x\|^2\} = \mathrm{Tr}\{\mathbf{R}_x\} < \infty$. Furthermore, without loss of generality, we assume that the temporal and spectral centers of gravity, respectively defined as $\bar{t}_x \triangleq \frac{1}{\bar{E}_x}\int_t t\, r_x(t,t)\,dt = \frac{1}{\bar{E}_x}\mathbb{E}\big\{\int_t t\,|x(t)|^2\,dt\big\}$ and $\bar{f}_x \triangleq \frac{1}{\bar{E}_x}\int_f f\, r_X(f,f)\,df = \frac{1}{\bar{E}_x}\mathbb{E}\big\{\int_f f\,|X(f)|^2\,df\big\}$ (with $r_X(f,f') = \mathbb{E}\{X(f)\,X^*(f')\}$), are zero. (This assumption can can always be met by considering the TF shifted process $x(t+\bar{t}_x)\,e^{-j2\pi\bar{f}_x t}$.) Several times, we will refer to the eigenvalue (or, Karhunen-Loéve) decomposition of the correlation operator, $\mathbf{R}_x = \sum_{k=1}^{\infty}\lambda_k\, u_k \otimes u_k^*$. Here, the eigenvalues $\lambda_k \geq 0$ are assumed to be sorted in decreasing order and the outer product $u_k \otimes u_k^*$ corresponds to a rank one operator with kernel $u_k(t)\,u_k^*(t')$.

## 2. UNCERTAINTY RELATIONS FOR NONSTATIONARY RANDOM PROCESSES

**Heisenberg-Type Inequalities.** We first derive bounds on the product of mean duration $\bar{T}_x$ and mean bandwidth $\bar{F}_x$ which are defined, respectively, via

$$\bar{T}_x^2 \triangleq \frac{\int_t t^2\, r_x(t,t)\,dt}{\int_t r_x(t,t)\,dt} = \frac{1}{\bar{E}_x}\mathbb{E}\Big\{\int_t t^2\,|x^2(t)|^2\,dt\Big\},$$

$$\bar{F}_x^2 \triangleq \frac{\int_f f^2\, r_X(f,f)\,df}{\int_f r_X(f,f)\,df} = \frac{1}{\bar{E}_x}\mathbb{E}\Big\{\int_f f^2\,|X(f)|^2\,df\Big\}.$$

Furthermore, we consider the mean "TF radius"

$$\bar{\rho}_x^2 \triangleq \Big(\frac{\bar{T}_x}{T}\Big)^2 + (T\bar{F}_x)^2,$$

with $T$ a normalization constant. The following theorem (which is based on results in [11]), relates the mean duration and bandwidth of a process to the eigenvalue spread of its correlation operator.

**Theorem 1.** *For any finite-energy random process, the mean TF radius $\bar{\rho}_x$ and the mean duration-bandwidth product $\bar{T}_x\bar{F}_x$ are bounded from below as*

$$\bar{\rho}_x^2 \geq \frac{1}{\pi}\Big(\Lambda_x^{(1)} - \frac{1}{2}\Big), \qquad \bar{T}_x\bar{F}_x \geq \frac{1}{2\pi}\Big(\Lambda_x^{(1)} - \frac{1}{2}\Big). \tag{1}$$

*Here,*

$$\Lambda_x^{(1)} \triangleq \frac{\sum_{k=1}^{\infty} k\lambda_k}{\sum_{k=1}^{\infty}\lambda_k}, \tag{2}$$

*with $\lambda_k$ the eigenvalues of $\mathbf{R}_x$. Equality in* (1) *is achieved iff the eigenfunctions of $\mathbf{R}_x$ equal the Hermite functions $h_k(t)$ [9–11],*

$$u_k(t) = h_k(t) \triangleq \frac{1}{\sqrt{2^{k-3/2}(k-1)!\,T}}\,H_{k-1}\Big(\sqrt{2\pi}\frac{t}{T}\Big)\,e^{-\pi(t/T)^2},$$

*with the Hermite polynomials $H_k(\eta) = (-1)^k e^{\eta^2}\frac{d^n}{d\eta^n}e^{-\eta^2}$.*

The parameter $\Lambda_x^{(1)}$ in the above theorem can be viewed as a measure of the eigenvalue spread, i.e., as effective rank, of the correlation operator $\mathbf{R}_x$. Hence, Theorem 1 shows that the mean TF radius $\bar{\rho}_x$ and the mean duration-bandwidth product $\bar{T}_x\bar{F}_x$ of a random process $x(t)$ are lower bounded in terms of the effective rank of $\mathbf{R}_x$. Note that due to $\sum_{k=1}^{\infty} k\lambda_k \geq \sum_{k=1}^{\infty}\lambda_k$, $\Lambda_x^{(1)} \geq 1$, which yields inequalities that are less tight but independent of the $\lambda_k$:

$$\bar{\rho}_x^2 \geq \frac{1}{2\pi}, \qquad \bar{T}_x\bar{F}_x \geq \frac{1}{4\pi}. \tag{3}$$

Since $\Lambda_x^{(1)} = 1$ iff $x(t)$ is an "almost deterministic" process, i.e., iff $x(t) = c\,u_1(t)$, $\mathbb{E}\{|c|^2\} = \bar{E}_x$, with rank-one correlation operator $\mathbf{R}_x = \bar{E}_x u_1 \otimes u_1^*$, (3) is an accordance with the known Heisenberg inequalities for deterministic signals.

The effective rank $\Lambda_x^{(1)}$ can also be interpreted as the effective number of uncorrelated components in $x(t)$ and thus as a measure of the "randomness" of $x(t)$. Thus, (1) also suggests that the mean TF radius $\bar{\rho}_x^2$ and the mean duration-bandwidth product $\bar{T}_x\bar{F}_x$ are larger for processes featuring less correlations. To support this claim, note that $r_x(t,t) = \int_\nu \bar{A}_x^{(\alpha)}(0,\nu)\, e^{j2\pi\nu t}\, d\nu$ and $r_X(f,f) = \int_\nu \bar{A}_x(\tau,0)\, e^{j2-\pi\tau f}\, d\tau$; here, $\bar{A}_x(\tau,\nu) \triangleq \mathbb{E}\{\langle x, \mathbf{S}_{\tau,\nu}x\rangle\}$ (with the TF shift operator $(\mathbf{S}_{\tau,\nu}x)(t) = x(t-\tau)\, e^{j2\pi\nu t}$) is the *expected ambiguity function* [14, 16, 21] of $x(t)$ that can be interpreted as a TF correlation function. Thus, the above Fourier relations show that a small temporal correlation width (i.e., small extension of $\bar{A}_x^{(\alpha)}(\tau,\nu)$ with respect to time lag $\tau$) leads to a large bandwidth $\bar{F}_x$ of $r_X(f,f) = \mathbb{E}\{|X(f)|^2\}$, and a small spectral correlation width (i.e., small extension of $\bar{A}_x^{(\alpha)}(\tau,\nu)$ with respect to frequency lag $\nu$) leads to a large duration $\bar{T}_x$ of $r_x(t,t) = \mathbb{E}\{|x(t)|^2\}$. Hence, processes with small TF correlations have larger duration and bandwidth.

**Uncertainty Relations Based on SPL Theory.** We next investigate how well a random process can be simultaneously supported in compact time and frequency intervals. Let $\mathbf{P}_{\mathcal{T}}$ an $\mathbf{P}_{\mathcal{F}}$ denote the orthogonal projection operators associated to the time interval $\mathcal{T} = [-T/2, T/2]$ and the frequency band $\mathcal{F} = [-F/2, F/2]$, respectively. A central role in SPL theory is played by the operator $\mathbf{Q}_{T,F} \triangleq \mathbf{P}_{\mathcal{F}}\mathbf{P}_{\mathcal{T}}\mathbf{P}_{\mathcal{F}}$ whose eigendecomposition is given by $\mathbf{Q}_{T,F} = \sum_{k=1}^{\infty}\mu_k\, p_k \otimes p_k^*$. Here, $p_k(t)$ denotes the $k$th prolate spheroidal wave function [4–7] and $\mu_k$ are the eigenvalues arranged in decreasing order. The $\mu_k$ have been shown to satisfy $0 < \mu_k < 1$; furthermore $\mu_k \approx 1$ for $1 \leq k \leq \lceil TF \rceil$ and $\mu_k \approx 0$ for $k > \lceil TF \rceil$ [5–7].

The mean energy concentration of $x(t)$ within $\mathcal{T}$ and $\mathcal{F}$ can be measured via the ratio of the mean energy of $(\mathbf{P}_{\mathcal{T}}\mathbf{P}_{\mathcal{F}}x)(t)$ and the mean energy of $x(t)$ itself. Correspondingly,

$$\bar{\sigma}_x^2 \triangleq 1 - \frac{\mathbb{E}\{\|\mathbf{P}_{\mathcal{F}}\mathbf{P}_{\mathcal{T}}x\|^2\}}{\mathbb{E}\{\|x\|^2\}} = \frac{\bar{E}_x - \mathbb{E}\{\|\mathbf{P}_{\mathcal{F}}\mathbf{P}_{\mathcal{T}}x\|^2\}}{\bar{E}_x}$$

measures the mean energy spread of $x(t)$ outside $\mathcal{T}$ and $\mathcal{F}$.

**Theorem 2.** *The mean energy spread $\bar{\sigma}_x^2$ is bounded from below as*

$$\bar{\sigma}_x^2 \geq \Lambda_x^{(2)} \triangleq \frac{\sum_{k=1}^{\infty}(1-\mu_k)\lambda_k}{\sum_{k=1}^{\infty}\lambda_k}, \tag{4}$$

*with equality iff the eigenfunctions of $\mathbf{R}_x$ equal the prolate spheroidal wave functions, i.e., iff $u_k(t) = p_k(t)$.*

It easily seen that $\Lambda_x^{(2)}$ is lower bounded as $\Lambda_x^{(2)} \geq 1-\mu_1 > 0$ and thus, no random process can be supported completely within the TF region $\mathcal{T}\times\mathcal{F} = [-T/2, T/2]\times[-F/2, F/2]$. Since $\Lambda_x^{(2)} = 1-\mu_1$ for a "semi-deterministic" process with rank-one correlation operator, (4) is consistent with the bound $\|\mathbf{P}_{\mathcal{F}}\mathbf{P}_{\mathcal{T}}x\|^2/\|x\|^2 \leq \mu_1$ for deterministic signals. Furthermore, due to $\mu_k \approx 1$ for $1 \leq k \leq \lceil TF \rceil$ and $\mu_k \approx 0$ for $k > \lceil TF \rceil$, we have $1-\mu_k \approx 0$ for $1 \leq k \leq \lceil TF \rceil$ and $1-\mu_k \approx 1$ for $k > \lceil TF \rceil$. Hence, $\Lambda_x^{(2)}$ is seen to be a measure of the eigenvalue spread of $\mathbf{R}_x$ relative to the area $TF$ of the underlying TF region $\mathcal{T}\times\mathcal{F}$. In particular,

$$\Lambda_x^{(2)} \approx \frac{\sum_{k>\lceil TF \rceil}\lambda_k}{\sum_k \lambda_k},$$

i.e., $\Lambda_x^{(2)}$ approximately measures the spread of the eigenvalues of $\mathbf{R}_x$ beyond $\lceil TF \rceil$. We conclude that the energy concentration of any nonstationary random process within $\mathcal{T}\times\mathcal{F}$ is less than the fraction of energy contained in the $\lceil TF \rceil$ largest eigenvalues of $\mathbf{R}_x$.

An alternative view can be obtained by investigating the restrictions regarding the simultaneous concentration of $x(t)$ within $\mathcal{T}$ and $\mathcal{F}$, respectively, as measured by the mean energy fractions

$$\bar{\kappa}_T^2 \triangleq \frac{1}{\bar{E}_x}\mathbb{E}\{\|\mathbf{P}_{\mathcal{T}}x\|^2\} = \frac{1}{\bar{E}_x}\mathbb{E}\Big\{\int_{-T/2}^{T/2}|x(t)|^2 dt\Big\}$$

$$\bar{\kappa}_F^2 \triangleq \frac{1}{\bar{E}_x}\mathbb{E}\{\|\mathbf{P}_{\mathcal{F}}x\|^2\} = \frac{1}{\bar{E}_x}\mathbb{E}\Big\{\int_{-F/2}^{F/2}|X(f)|^2 df\Big\}$$

Here, it is possible to prove the following result.

**Theorem 3.** *Any finite energy random process can have mean energy fractions $\bar{\kappa}_T^2$ and $\bar{\kappa}_F^2$ within $\mathcal{T}$ and $\mathcal{F}$, respectively, iff*

$$\arccos(\bar{\kappa}_T) + \arccos(\bar{\kappa}_F) \geq \arccos\Big(\sqrt{1-\Lambda_x^{(2)}}\Big) \tag{5}$$

*with $\Lambda_x^{(2)}$ as defined in* (4).

Since (5) places an upper bound on the possible values of $\bar{\kappa}_T^2$ and $\bar{\kappa}_F^2$ that can occur simultaneously, the above theorem shows that any random process cannot be arbitrarily concentrated in the time interval $\mathcal{T}$ and the frequency band $\mathcal{F}$ simultaneously. In particular, the possible values of $\bar{\kappa}_T^2$ and $\bar{\kappa}_F^2$ are again determined by the parameter $\Lambda_x^{(2)}$ that measures the eigenvalue spread of $\mathbf{R}_x$ relative to $TF$. The pairs $(\bar{\kappa}_T^2, \bar{\kappa}_F^2)$ which are achievable correspond to the region below the elliptical curve defined by (5) (see Section 4).

We note that it is again possible to interpret Theorems 2 and 3 in terms of the temporal and spectral correlation, with the conclusion that the energy of less correlated processes is more spread out in time and/or frequency.

## 3. CONCENTRATION OF TF ENERGY SPECTRA

We next develop concentration inequalities for TF energy spectra which parallel the uncertainty inequalities for random processes derived previously. We first briefly review the required TF concepts (more details can be found in [7, 14, 16]).

**TF Prerequisites.** One class of time-varying power spectra, type I TF energy spectra, are defined as[1] [7, 14–16]

$$P_x^{(\mathrm{I})}(t,f;\mathbf{C}) \triangleq \langle \mathbf{R}_x, \mathbf{C}_{t,f}\rangle.$$

Here, $\mathbf{C} = \sum_{k=1}^{\infty}\gamma_k\, c_k \otimes c_k^*$ is a suitably chosen trace-normalized ($\mathrm{Tr}\{\mathbf{C}\} = \sum_{k=1}^{\infty}\gamma_k = 1$) "TF localization" operator. Furthermore, $\mathbf{C}_{t,f} = \mathbf{S}_{t,f}\mathbf{C}\mathbf{S}_{t,f}^+$ is a TF shifted version of $\mathbf{C}$. Important type I spectra are the generalized Wigner-Ville spectrum (GWVS) $\overline{W}_x^{(\alpha)}(t,f)$ [7, 14], $\alpha \in \mathbb{R}$, (which includes the Wigner-Ville spectrum $\overline{W}_x^{(0)}(t,f)$ [7, 14, 17] and the Rihaczek spectrum [7, 14, 18] $\overline{W}_x^{(1/2)}(t,f)$ as special cases), Page's instantaneous power spectrum [19], Levin's spectrum [23], and the physical spectrum [20] $P_x^{(\mathrm{I})}(t,f;g\otimes g^*)$ (with $g(t)$ the underlying window).

[1] The inner product of two operators is defined as $\langle \mathbf{H}_1, \mathbf{H}_2\rangle \triangleq \mathrm{Tr}\{\mathbf{H}_1\mathbf{H}_2^+\}$ (with $\mathbf{H}^+$ denoting the *adjoint* of $\mathbf{H}$ [13]).

An alternative to type I spectra are type II TF energy spectra [14, 16]. Their definition is based on the innovations system representation $x(t) = (\mathbf{H}_x n)(t)$ of $x(t)$, where $n(t)$ denotes normalized stationary white noise and $\mathbf{H}_x$ is a (non-unique) innovations system defined by the condition $\mathbf{H}_x\mathbf{H}_x^+ = \mathbf{R}_x$. Type II spectra are then defined as [14]

$$P_x^{(\mathrm{II})}(t,f;\mathbf{C}) \triangleq \left|\left\langle \mathbf{H}_x, \mathbf{C}_{t,f}\right\rangle\right|^2.$$

The most important type II spectrum is the generalized evolutionary spectrum (GES) $G_x^{(\alpha)}(t,f)$ [21] ($\alpha \in \mathbb{R}$), with the special cases Weyl spectrum $G_x^{(0)}(t,f)$ [21], evolutionary spectrum $G_x^{(1/2)}(t,f)$ [3], and transitory evolutionary spectrum $G_x^{(-1/2)}(t,f)$ [21, 22].

We finally recall the *underspread* concept for linear systems (operators) and random processes To this end, consider the following moments of the *spreading function* $S_{\mathbf{H}}(\tau,\nu)$ [16, 24] of a linear system (or operator) $\mathbf{H}$:

$$\tau_{\mathbf{H}}^2 = \frac{\int_\tau\int_\nu |\tau|^2 |S_{\mathbf{H}}(\tau,\nu)|^2 d\tau\, d\nu}{\int_\tau\int_\nu |S_{\mathbf{H}}(\tau,\nu)|^2 d\tau\, d\nu}, \quad \nu_{\mathbf{H}}^2 = \frac{\int_\tau\int_\nu |\nu|^2 |S_{\mathbf{H}}(\tau,\nu)|^2 d\tau\, d\nu}{\int_\tau\int_\nu |S_{\mathbf{H}}(\tau,\nu)|^2 d\tau\, d\nu}.$$

Since $S_{\mathbf{H}}(\tau,\nu)$ characterizes the TF shifts of $\mathbf{H}$, $\tau_{\mathbf{H}}$ and $\nu_{\mathbf{H}}$ are global measures of the amount of delay (time-shift) and Doppler (frequency shift), respectively, introduced by $\mathbf{H}$. A system $\mathbf{H}$ is then called underspread if $\tau_{\mathbf{H}}\nu_{\mathbf{H}} \ll 1$, meaning that $\mathbf{H}$ introduces only small TF shifts. In the case $\mathbf{H} = \mathbf{R}_x$, we simply write $\tau_x = \tau_{\mathbf{R}_x}$ and $\nu_x = \nu_{\mathbf{R}_x}$, with the interpretation that $\tau_x$ and $\nu_x$ characterize the amount of temporal and spectral correlations of $x(t)$, respectively (this interpretation is supported by the fact that $S_{\mathbf{R}_x}(\tau,\nu) = \bar{A}_x(\tau,\nu)$). A process $x(t)$ is called underspread if $\tau_x\nu_x \ll 1$, i.e., if $x(t)$ features only small TF correlations.

**Concentration Inequalities for Type I Spectra.** Let us measure the TF spread (or TF concentration) of a TF function $M(t,f)$ via the TF radius

$$\rho^2(M) \triangleq \frac{\int_t\int_f \left[\left(\frac{t}{T}\right)^2 + (Tf)^2\right] M(t,f)\, dt\, df}{\int_t\int_f M(t,f)\, dt\, df}.$$

Our first result applies to a particular subclass of type I spectra and follows straightforwardly from Theorem 1.

**Corollary 4.** *Assume $P_x^{(\mathrm{I})}(t,f;\mathbf{C})$ satisfies the marginal properties,*

$$\int_t P_x^{(\mathrm{I})}(t,f;\mathbf{C})\, dt = \mathbb{E}\{|X(f)|^2\}, \quad \int_f P_x^{(\mathrm{I})}(t,f;\mathbf{C})\, df = \mathbb{E}\{|x(t)|^2\}.$$

*Then, for any finite-energy random process $x(t)$ there is*

$$\rho^2\left(P_x^{(\mathrm{I})}\right) = \bar{\rho}_x^2 \geq \frac{1}{\pi}\left(\Lambda_x^{(1)} - \frac{1}{2}\right), \tag{6}$$

*with $\Lambda_x^{(1)}$ as defined in* (2). *Equality in* (6) *holds iff the eigenfunctions of $\mathbf{R}_x$ equal the Hermite functions, i.e., iff $u_k(t) = h_k(t)$.*

Important type I spectra satisfying the marginal properties are the GWVS (which includes the Wigner-Ville spectrum and Rihaczek spectrum as special cases), Page's instantaneous power spectrum, and Levin's spectrum. According to the foregoing corollary, the TF radius $\rho^2\left(P_x^{(\mathrm{I})}\right)$ of these spectra is lower bounded in terms of the eigenvalue spread parameter $\Lambda_x^{(1)}$. This indicates that type I spectra satisfying the marginal properties cannot be too much concentrated. We caution, however, that in general the above mentioned type I spectra can assume negative and/or complex values and thus the interpretation of $\rho^2\left(P_x^{(\mathrm{I})}\right)$ as a measure of the TF spread of $P_x^{(\mathrm{I})}(t,f;\mathbf{C})$ has to be justified properly in each case. This is not necessary if the type I spectrum is positive (in the following "positive" more precisely means non-negative), i.e., if $\mathbf{C} \geq \mathbf{0}$.

**Theorem 5.** *For any positive type I spectrum $P_x^{(\mathrm{I})}(t,f;\mathbf{C})$ and any finite-energy random process $x(t)$,*

$$\rho^2\left(P_x^{(\mathrm{I})}\right) \geq \frac{1}{\pi}\left(\Lambda_x^{(1)} + \Lambda_{\mathbf{C}} - 1\right) \tag{7}$$

*with $\Lambda_{\mathbf{C}} = \sum_{k=1}^{\infty} k\gamma_k$. Equality in* (7) *is achieved iff the eigenfunctions of* $\mathbf{R}_x$ and $\mathbf{C}$ *are the Hermite functions, $u_k(t) = c_k(t) = h_k(t)$.*

Thus, as compared to type I spectra satisfying the marginal properties, positive type I spectra have a TF radius $\rho^2\left(P_x^{(\mathrm{I})}\right)$ that essentially is increased by the eigenvalue spread $\Lambda_{\mathbf{C}}$ of the underlying TF localization operator $\mathbf{C}$. Note that here $\rho^2\left(P_x^{(\mathrm{I})}\right)$ is a proper measure of the TF spread of $P_x^{(\mathrm{I})}(t,f;\mathbf{C}) \geq 0$. Obviously, $\Lambda_{\mathbf{C}} \geq 1$ with $\Lambda_{\mathbf{C}} = 1$ for a rank one TF localization operator $\mathbf{C} = g \otimes g^*$. Since rank one operators $\mathbf{C} = g \otimes g^*$ correspond to the physical spectrum $P_x^{(\mathrm{I})}(t,f;g\otimes g^*)$ (with window $g(t)$), we conclude that the physical spectrum features the best TF concentration among all positive type I spectra. In that case, $\rho^2\left(P_x^{(\mathrm{I})}\right) \geq \frac{\Lambda_x^{(1)}}{\pi} \geq \frac{1}{\pi}$.

We next restrict to the GWVS $\overline{W}_x^{(\alpha)}(t,f)$ as a particularly important subclass of type I spectra. Since $\overline{W}_x^{(\alpha)}(t,f)$ is not necessarily real-valued and positive, we consider the alternative TF concentration measure $\tilde{\rho}^2\left(\overline{W}_x^{(\alpha)}\right)$ defined as

$$\tilde{\rho}^2\left(\overline{W}_x^{(\alpha)}\right) \triangleq \frac{\int_t\int_f \left[\left(\frac{t}{T}\right)^2 + (Tf)^2\right] \left|\overline{W}_x^{(\alpha)}(t,f)\right|^2 dt\, df}{\int_t\int_f \left|\overline{W}_x^{(\alpha)}(t,f)\right|^2 dt\, df}$$

Inequalities for $\tilde{\rho}^2\left(\overline{W}_x^{(\alpha)}\right)$ are slightly more difficult to establish. The following theorem builds upon results from [11] and [25] and shows that the Wigner-Ville spectrum, i.e., the GWVS with $\alpha = 0$, has maximum TF concentration (minimum TF radius).

**Theorem 6.** *For any finite energy process $x(t)$, the TF radius of the GWVS satisfies*

$$\tilde{\rho}^2\left(\overline{W}_x^{(\alpha)}\right) = \tilde{\rho}^2\left(\overline{W}_x^{(0)}\right) + \alpha^2\left[\left(\frac{\tau_x}{T}\right)^2 + (\nu_x T)^2\right] \tag{8}$$

$$\geq \frac{1}{2\pi}\left(\Lambda_x^{(3)} - \frac{1}{2}\right) + 2\alpha^2\tau_x\nu_x, \tag{9}$$

*with*

$$\Lambda_x^{(3)} \triangleq \frac{\sum_{k=1}^{\infty} k\lambda_k^2}{\sum_{k=1}^{\infty}\lambda_k^2}.$$

The foregoing theorem shows two things. First, from (8) it is seen that the Wigner-Ville spectrum, i.e., the GWVS with $\alpha = 0$, has maximum TF concentration within the entire GWVS family. Only in the case of processes with small moments $\tau_x$ and $\nu_x$, i.e., for underspread processes, will the TF concentration of the other GWVS members be nearly as good as that of the Wigner-Ville spectrum, i.e., $\tilde{\rho}_x^2\left(\overline{W}_x^{(\alpha)}\right) \approx \tilde{\rho}_x^2\left(\overline{W}_x^{(0)}\right)$. Second, according to (9), the TF concentration of the GWVS is bounded from below, with the lower bound being determined by $\Lambda_x^{(3)}$ and the product $\tau_x\nu_x$. The parameter $\Lambda_x^{(3)}$ again is a measure of the eigenvalue spread of the correlation operator $\mathbf{R}_x$ and hence can be interpreted as another definition of an "effective rank" of $\mathbf{R}_x$. We note that $\alpha = 0$ yields the smallest lower bound in (9), i.e.,

$$\tilde{\rho}_x^2\left(\overline{W}_x^{(0)}\right) \geq \frac{1}{2\pi}\left(\Lambda_x^{(3)} - \frac{1}{2}\right).$$

Since $\Lambda_x^{(3)} \geq 1$, $\tilde{\rho}_x^2\left(\overline{W}_x^{(0)}\right) \geq \frac{1}{4\pi}$ for any finite-energy process, independently of the KL eigenvalues $\lambda_k$.

From the uncertainty relations derived above, we can conclude an important general rule: for greater effective rank (broader KL eigenvalue spectrum) of the correlation operator $\mathbf{R}_x$, the TF spread of type I spectra will be larger, i.e., $P_x^{(\mathrm{I})}(t,f;\mathbf{C})$ will have a larger TF support region. A larger effective rank typically occurs for underspread processes. Here, the "spread increasing" terms $\Lambda_{\mathbf{C}}$ and $2\alpha^2\tau_x\nu_x$ in (7) and (9), respectively, become less significant.

**Concentration Inequalities for Type II Spectra.** Type II TF energy spectra are much more difficult to treat from a theoretical point of view than type I spectra. We thus restrict to the practically most important case of the GES. We furthermore assume that the innovations system $\mathbf{H}_x$ is chosen as the positive semi-definite root of $\mathbf{R}_x$, i.e., $\mathbf{H}_x = \mathbf{R}_x^{1/2} = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, u_k \otimes u_k^*$. The following theorem was obtained by adapting and combining results from [11] and [25].

**Theorem 7.** *For any finite-energy process $x(t)$, the TF radius of the GES (defined with the positive semi-definite innovations system $\mathbf{H}_x$) satisfies*

$$\rho_x^2(G_x^{(\alpha)}) = \bar{\rho}_x^2(G_x^{(0)}) + \alpha^2\left[\left(\frac{\tau_{\mathbf{H}_x}}{T}\right)^2 + (T\nu_{\mathbf{H}_x})^2\right] \qquad (10)$$

$$\geq \frac{1}{2\pi}\left(\Lambda_x^{(1)} - \frac{1}{2}\right) + 2\alpha^2\tau_{\mathbf{H}_x}\nu_{\mathbf{H}_x}, \qquad (11)$$

*with $\Lambda_x^{(1)}$ as defined in* (2).

This theorem shows several facts. First, according to (10) the TF spread of the GES (using the positive semi-definite innovations system) is minimum for $\alpha = 0$, i.e., for the Weyl spectrum. For $\alpha \neq 0$, the TF radius will be larger by an amount determined by the moments $\tau_{\mathbf{H}_x}$ and $\nu_{\mathbf{H}_x}$. For an underspread process, the innovations systems $\mathbf{H}_x$ is underspread, too, i.e., $\tau_{\mathbf{H}_x}$ and $\nu_{\mathbf{H}_x}$ are small. In that case, the TF concentration of the GES with $\alpha \neq 0$ is comparable to that of the GES with $\alpha = 0$. Second, (11) shows that $\rho^2(G_x^{(\alpha)})$ is bounded from below with the lower bound determined by the eigenvalue spread (effective rank) $\Lambda_x^{(1)}$ of $\mathbf{R}_x$ and the displacement moments of $\mathbf{H}_x$ (note that $\Lambda_x^{(1)}$ also measures the eigenvalue spread of $\mathbf{H}_x$). Furthermore, since $\Lambda_x^{(1)} \geq 1$ (with $\Lambda_x^{(1)} = 1$ for a rank-one correlation operator), the smallest bound in (11) is $\bar{\rho}_x^2(G_x^{(0)}) \geq \frac{1}{4\pi}$, which is independent of the KL eigenvalues $\lambda_k$.

We conclude that the TF concentration of the GES depends on the effective rank $\Lambda_x^{(1)}$ of $\mathbf{R}_x$ and $\mathbf{H}_x$ in a way such that the TF support region of the GES is larger for processes whose correlation operator has a large eigenvalue spread. A larger effective rank typically occurs in the case of an underspread process that features small TF correlations. In that case, the term $\tau_{\mathbf{H}_x}\nu_{\mathbf{H}_x}$ in (11) that increases the lower bound on the TF radius will be negligibly small.

## 4. NUMERICAL SIMULATIONS

We consider an underspread process $x(t)$ and an overspread process $\tilde{x}(t)$. Both processes consist of the same components with equal mean powers, only that the components of $\tilde{x}(t)$ are correlated.

The mean time-bandwidth products of these processes were $\bar{T}_x\bar{F}_x = 1.98$ and $\bar{T}_{\tilde{x}}\bar{F}_{\tilde{x}} = 1.97$. With effective ranks of $\Lambda_x^{(1)} = 12.8$ and $\Lambda_{\tilde{x}}^{(1)} = 9.3$, the corresponding lower bounds in (1) were obtained as 1.96 and 1.41, respectively. Similarly, for $TF = 10$, we obtained energy spreads of $\bar{\sigma}_x^2 = 0.54$ and $\bar{\sigma}_{\tilde{x}}^2 = 0.52$, which indeed are larger than the corresponding lower bounds $\Lambda_x^{(2)} = 0.52$ and $\Lambda_{\tilde{x}}^{(2)} = 0.36$, respectively, in (4). It is seen that (1) and (4) are quite tight in the underspread case. This is due to the fact that both the Hermite functions and the prolate spheroidal wave functions are "approximate" eigenfunctions of $\mathbf{R}_x$. Furthermore, the overspread

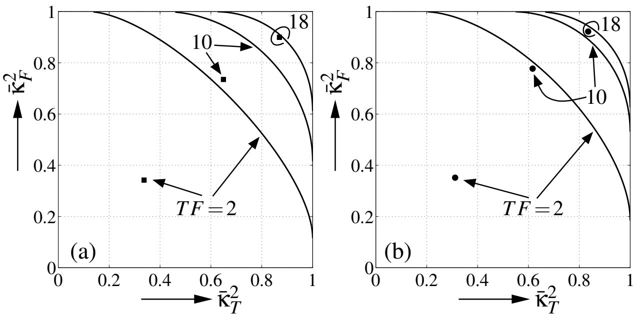


**Figure 1**: *Allowed regions for energy concentration pairs $(\kappa_T^2, \kappa_F^2)$ for (a) an underspread process and (b) an overspread process.*

process features slightly better energy concentration (smaller $\bar{T}_{\tilde{x}}\bar{F}_{\tilde{x}}$ and $\bar{\sigma}_{\tilde{x}}^2$) than the overspread process, verifying our claim that more correlated processes are less spread out in time and/or frequency.

We furthermore determined $\Lambda_x^{(2)}$ and $\Lambda_{\tilde{x}}^{(2)}$ for $TF \in \{2, 10, 18\}$. Using (5), this lead to the elliptical boundary curves shown in Fig. 1. The possible energy concentration pairs have to lie beneath these curves. The actual energy concentrations $(\bar{\kappa}_T^2, \bar{\kappa}_F^2)$ of $x(t)$ and $\tilde{x}(t)$ are indicated by square and circles, respectively. It is seen that the boundary curves for the overspread process are above those of the underspread process (particularly for small $TF$), verifying that energy concentration can be better in the overspread case.

Finally, several TF energy spectra of $x(t)$ and $\tilde{x}(t)$ are shown in Fig. 2 (conforming with [14, 16], the spectra are indeed approximately equivalent in the underspread case). For Wigner-Ville and Rihaczek spectrum,we obtained TF radii $\tilde{\rho}^2\big(\overline{W}_x^{(0)}\big) \approx \tilde{\rho}^2\big(\overline{W}_x^{(1/2)}\big) = 1.43$ in the underspread case; for both spectra, the corresponding lower bound in (9) (obtained with $\Lambda_x^{(3)} = 9.1$) is 1.36 (note that $\alpha^2\tau_x\nu_x$ is negligible). For the overspread process, $\tilde{\rho}^2\big(\overline{W}_{\tilde{x}}^{(0)}\big) = 1.17$ and $\tilde{\rho}^2\big(\overline{W}_{\tilde{x}}^{(1/2)}\big) = 1.26$ with corresponding lower bounds 0.82 and 1.17, respectively (here, $\Lambda_{\tilde{x}}^{(3)} = 5.6$). For the physical spectrum, $\rho^2\big(P_x^{(\mathrm{I})}\big) = 2.11$ and $\rho^2\big(P_{\tilde{x}}^{(\mathrm{I})}\big) = 1.94$ with respective lower bounds of 2.03 and 1.49. For the Weyl and evolutionary spectrum, the TF radii were $\rho^2\big(G_x^{(0)}\big) = 2.01$ and $\rho^2\big(G_x^{(1/2)}\big) = 2.04$ (with respective lower bound of 1.96 in (11); note that $\tau_{\mathbf{H}_x}\nu_{\mathbf{H}_x}$ is negligible) and $\rho^2\big(G_{\tilde{x}}^{(0)}\big) = 1.79$ and $\rho^2\big(G_{\tilde{x}}^{(1/2)}\big) = 1.87$ (the lower bounds in (11) are 1.41 and 1.8). In summary, Wigner-Ville and Rihaczek spectrum as well as Weyl and evolutionary spectrum are equally well concentrated in the underspread case. In contrast, in the case of the overspread process, $\alpha = 0$ is preferable to $\alpha = 1/2$ with regard to TF energy concentration.

## 5. CONCLUSIONS

We presented several uncertainty and concentration inequalities for nonstationary random process and time-frequency (TF) energy spectra (time-varying power spectra). These inequalities provided several insights: random processes cannot be arbitrarily concentrated in time and frequency simultaneously and the the lower bounds on the achievable energy concentration are determined by the the effective rank of the correlation operator. Furthermore, underspread (i.e., less correlated) processes typically have poorer energy concentration properties. In a similar way, the TF concentration of TF energy spectra is bounded from below in terms of the effective rank of $\mathbf{R}_x$. While the TF concentration of all TF energy spectra is essentially the same for underspread processes, Wigner-Ville spectrum and Weyl spectrum feature the best TF concentration in the case of overspread processes.

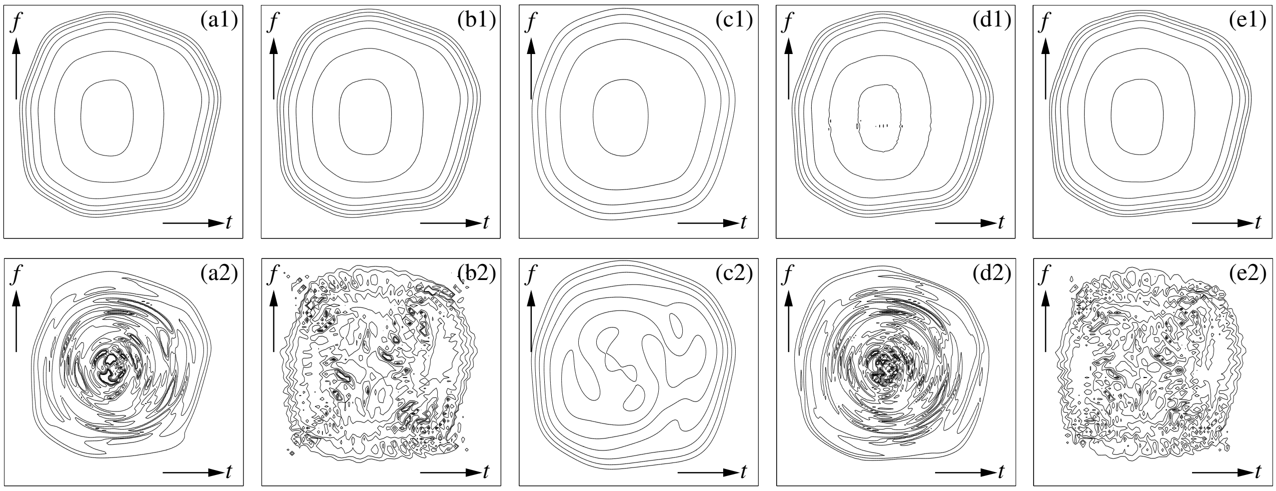


**Figure 2**: *(a1) Wigner-Ville spectrum* $\overline{W}_x^{(0)}(t,f)$*, (b1) Rihaczek spectrum* $\overline{W}_x^{(1/2)}(t,f)$*, (c1) physical spectrum* $P_x^{(\mathrm{I})}(t,f;g\otimes g^*)$*, (d1) Weyl spectrum* $G_x^{(0)}(t,f)$*, (e1) and evolutionary spectrum* $G_x^{(1/2)}(t,f)$ *of an underspread process; (a2)–(e2) same for an overspread process.*

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