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TIME-FREQUENCY FORMULATION AND DESIGN OF OPTIMAL DETECTORS*

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Abstract—We present time-frequency (TF) formulations of optimal detectors for various nonstationary detection scenarios involving underspread processes. These TF formulations yield a better understanding of optimal detectors and simple TF design procedures.

1 INTRODUCTION

We consider the detection of a signal $s(t)$ corrupted by nonstationary noise $n(t)$. This can be formulated as a hypothesis test H_0: $r(t) = n(t)$ versus H_1: $r(t) = s(t) + n(t)$, where $r(t)$ is the observed signal. The noise is assumed to be Gaussian, zero-mean, and circular complex with correlation function $R_n(t,t') = \mathrm{E}\{n(t)\, n^*(t')\}$ or, equivalently, correlation operator[1] $\mathbf{R}_n$. We note that $n(t) \in \mathcal{S}_n$ where the *noise space* $\mathcal{S}_n = \mathcal{R}(\mathbf{R}_n)$ is the range of $\mathbf{R}_n$.

It is well known that the optimal detectors use the observed signal $r(t)$ to form a (sufficient) test statistic $\Lambda(r)$ which is compared to a threshold to obtain the actual decision [1]-[3]. We shall distinguish the following two cases.

Case 1: Deterministic signal. The signal $s(t)$ is modeled as deterministic and known. We assume that $s(t) \in \mathcal{S}_n$ since otherwise perfect detection would be possible [2]. Hence, there is also $r(t) \in \mathcal{S}_n$, so that the noise space $\mathcal{S}_n$ is simultaneously the *observation space.* The optimal detector in this case is the *likelihood ratio detector* [1]-[3] whose test statistic is the linear functional[2]

$$\Lambda_o(r) = \mathrm{Re}\langle \mathbf{R}_n^{-1} r, s\rangle. \tag{1}$$

The performance of this detector is completely characterized by its *deflection*[3] [1, 2]

$$d_o^2 = 2\,\langle \mathbf{R}_n^{-1} s, s\rangle\,. \tag{2}$$

Case 2: Random signal. The signal $s(t)$ is a nonstationary, zero-mean, circular complex random process with correlation operator $\mathbf{R}_s$. Furthermore, $s(t)$ and $n(t)$ are uncorrelated. To exclude perfect detection, we assume $\mathcal{S}_s \subseteq \mathcal{S}_n$ where $\mathcal{S}_s = \mathcal{R}(\mathbf{R}_s)$ is the signal space.

There exist two important optimal detectors. The *likelihood ratio detector* assumes $s(t)$ to be Gaussian; the corresponding test statistic is the quadratic form [1]-[3]

$$\Lambda_l(r) = \langle \mathbf{H}_l r, r\rangle \tag{3}$$

with the operator $\mathbf{H}_l$ given by

$$\mathbf{H}_l = \mathbf{R}_n^{-1} - (\mathbf{R}_s + \mathbf{R}_n)^{-1} = \mathbf{R}_n^{-1}\mathbf{R}_s(\mathbf{R}_s + \mathbf{R}_n)^{-1}, \tag{4}$$

or $\mathbf{H}_l = \mathbf{R}_n^{-1}\mathbf{H}_{\mathrm{opt}}$ where $\mathbf{H}_{\mathrm{opt}} = \mathbf{R}_s(\mathbf{R}_s + \mathbf{R}_n)^{-1}$ is the *nonstationary Wiener filter* [1, 5]. The second optimal test statistic, defined as the quadratic test statistic maximizing the deflection, is given by the quadratic form [4]

$$\Lambda_d(r) = \langle \mathbf{H}_d r, r\rangle \qquad \text{with} \quad \mathbf{H}_d = \mathbf{R}_n^{-1}\mathbf{R}_s\mathbf{R}_n^{-1}. \tag{5}$$

This detector does not assume the signal to be the Gaussian. The (maximum) deflection achieved is given by[4]

$$d_d^2 = \mathrm{tr}\left\{\mathbf{\Gamma}^2\right\} \qquad \text{with} \quad \mathbf{\Gamma} = \mathbf{R}_n^{-1/2}\mathbf{R}_s\mathbf{R}_n^{-1/2}\,, \tag{6}$$

where $\mathbf{\Gamma}$ is known as *SNR operator* [3].

Stationary case. In the limiting (asymptotic) case of stationary processes, the test statistics (1), (3), and (5) and the deflections (2) and (6) can be expressed in the frequency domain as

$$\Lambda_o(r) = \int_f \frac{\mathrm{Re}\{R(f)\,S^*(f)\}}{P_n(f)}\,df, \quad d_o^2 = 2\int_f \frac{|S(f)|^2}{P_n(f)}\,df, \tag{7}$$

$$\Lambda_l(r) = \int_f \frac{P_s(f)}{P_n(f)\,[P_s(f) + P_n(f)]}\,|R(f)|^2\,df, \tag{8}$$

$$\Lambda_d(r) = \int_f \frac{P_s(f)}{P_n^2(f)}\,|R(f)|^2\,df, \quad d_d^2 = \int_f \left[\frac{P_s(f)}{P_n(f)}\right]^2 df, \tag{9}$$

where $R(f)$ and $S(f)$ are the Fourier transforms of $r(t)$ and $s(t)$, respectively, and $P_s(f)$ and $P_n(f)$ are the power spectral densities of $s(t)$ and $n(t)$, respectively. These frequency-domain expressions involve simple scalar products and inverses (instead of operator products and inverses) and are hence easily interpreted. They also allow a simple frequency-domain design of optimal detectors.

Outline of paper. Most previous papers on the subject (e.g., [6, 7]) have described exact time-frequency (TF) implementations of test statistics. In contrast, we propose approximate TF formulations which extend the simple frequency-domain expressions (7)-(9) obtained in the stationary case to the practically important class of "underspread" nonstationary processes [8, 9, 5]. Section 2 summarizes the TF tools required. Sections 3 and 4 discuss the TF formulation and TF design of the optimal detectors for a deterministic signal and for a random signal, respectively. Optimal signal design is reformulated as a TF signal synthesis problem. Section 5 extends the TF detector design to the case of a random signal with reduced *a priori* knowledge. The close-to-optimal performance of the TF designed detectors is verified using computer simulations.

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[1] The correlation operator $\mathbf{R}_n$ is the linear operator whose kernel is the correlation function $R_n(t,t')$.

[2] The inner product is defined as $\langle x,y\rangle = \int x(t)\,y^*(t)\,dt$. Integrals are from $-\infty$ to ∞ unless stated otherwise. All inverse operators (e.g., $\mathbf{R}_n^{-1}$) are to be understood as (pseudo-)inverses on the observation space $\mathcal{S}_n$.

[3] The deflection of a test statistic $\Lambda(r)$ is defined as $d^2 = \frac{(\mathrm{E}_1\{\Lambda\} - \mathrm{E}_0\{\Lambda\})^2}{\mathrm{var}_0\{\Lambda\}}$ where E_i and var_i are the conditional expectation and variance, respectively, under hypothesis H_i [1, 4].

[4] $\mathrm{tr}\{\cdot\}$ denotes the trace of an operator.

2 TIME-FREQUENCY FUNDAMENTALS

Weyl symbol and spreading function. Our TF formulations will be based on the *Weyl symbol* (WS). The WS of a linear operator (linear, time-varying system) $\mathbf{H}$ is a "time-varying frequency response" defined as [10]-[12]

$$L_{\mathbf{H}}(t,f) = \int_\tau H\Big(t+\frac{\tau}{2}, t-\frac{\tau}{2}\Big)\, e^{-j2\pi f\tau}\, d\tau$$

where $H(t,t')$ is the kernel (impulse response) of $\mathbf{H}$. This kernel can be re-obtained from $L_{\mathbf{H}}(t,f)$ according to

$$H(t,t') = \int_f L_{\mathbf{H}}\Big(\frac{t+t'}{2}, f\Big)\, e^{j2\pi f(t-t')}\, df\,. \qquad (10)$$

Using the WS, bilinear forms can be expressed as

$$\langle \mathbf{H}x, y\rangle = \langle L_{\mathbf{H}}, W_{y,x}\rangle = \int_t\!\int_f L_{\mathbf{H}}(t,f)\, W^*_{y,x}(t,f)\, dt\, df, \quad (11)$$

where $W_{y,x}(t,f) = \int_\tau y\big(t+\frac{\tau}{2}\big)\, x^*\big(t-\frac{\tau}{2}\big)\, e^{-j2\pi f\tau}\, d\tau$ is the cross Wigner distribution of $y(t)$ and $x(t)$ [13].

The *spreading function* of an operator $\mathbf{H}$,

$$S_{\mathbf{H}}(\tau,\nu) = \int_t H\Big(t+\frac{\tau}{2}, t-\frac{\tau}{2}\Big)\, e^{-j2\pi\nu t}\, dt\,, \qquad (12)$$

is the 2-D Fourier transform of the WS and can be shown to describe the TF displacements caused by $\mathbf{H}$ [14].

Wigner-Ville spectrum and expected ambiguity function. If $\mathbf{H} = \mathbf{R}_x$ is the correlation operator of a nonstationary random process $x(t)$, then its WS is known as the *Wigner-Ville spectrum* (WVS) of $x(t)$ [15],

$$\overline{W}_x(t,f) = L_{\mathbf{R}_x}(t,f) = \int_\tau R_x\Big(t+\frac{\tau}{2}, t-\frac{\tau}{2}\Big)\, e^{-j2\pi f\tau}\, d\tau.$$

The WVS describes the average TF energy distribution of $x(t)$ [15]. Furthermore, the spreading function of $\mathbf{R}_x$ equals the *expected ambiguity function* (EAF) of $x(t)$ [8],

$$\bar{A}_x(\tau,\nu) = S_{\mathbf{R}_x}(\tau,\nu) = \int_t R_x\Big(t+\frac{\tau}{2}, t-\frac{\tau}{2}\Big)\, e^{-j2\pi\nu t}\, dt, \quad (13)$$

which can be interpreted as a TF correlation function [8].

Underspread operators. A linear operator $\mathbf{H}$ is *underspread* if its spreading function $S_{\mathbf{H}}(\tau,\nu)$ is effectively supported within a *small* region $\mathcal{A}_{\mathbf{H}}$ about the origin of the (τ,ν)-plane [8, 9]. An underspread operator causes only small TF displacements. Two operators are *jointly underspread* if their spreading functions are effectively supported within the *same* small region about the origin. For jointly underspread operators $\mathbf{H}_1$, $\mathbf{H}_2$, it can be shown that [9]

$$L_{\mathbf{H}_1\mathbf{H}_2}(t,f) \approx L_{\mathbf{H}_1}(t,f)\, L_{\mathbf{H}_2}(t,f)\,.$$

Simulation results suggest that an underspread operator $\mathbf{H}$ is jointly underspread with its (pseudo-)inverse $\mathbf{H}^{-1}$, so that $L_{\mathbf{H}^{-1}\mathbf{H}}(t,f) \approx L_{\mathbf{H}^{-1}}(t,f)\, L_{\mathbf{H}}(t,f)$ in the TF region $\mathcal{G}_{\mathbf{H}}$ which corresponds to the range $\mathcal{S}_{\mathbf{H}} = \mathcal{R}(\mathbf{H})$ of $\mathbf{H}$ in the sense of [16]. Combining with $\mathbf{H}^{-1}\mathbf{H} = \mathbf{I}$, where $\mathbf{I}$ is the identity operator on $\mathcal{S}_{\mathbf{H}}$, we obtain $L_{\mathbf{H}^{-1}}(t,f)\, L_{\mathbf{H}}(t,f) \approx L_{\mathbf{I}}(t,f)$. It can furthermore be shown [16] that $L_{\mathbf{I}}(t,f) \approx 1$ on $\mathcal{G}_{\mathbf{H}}$, so that we finally obtain

$$L_{\mathbf{H}^{-1}}(t,f) \approx \frac{1}{L_{\mathbf{H}}(t,f)}, \quad (t,f) \in \mathcal{G}_{\mathbf{H}}. \qquad (14)$$

Underspread random processes. A nonstationary random process $x(t)$ is *underspread* if its correlation operator $\mathbf{R}_x$ is underspread, i.e., if its EAF $\bar{A}_x(\tau,\nu)$ is effectively supported within a *small* region $\mathcal{A}_x = \mathcal{A}_{\mathbf{R}_x}$ about the origin of the (τ,ν)-plane [8, 9]. This means that process components far apart in the TF plane are effectively uncorrelated. Two processes are *jointly underspread* if their EAFs are effectively supported within the *same* small region about the origin.

3 CASE 1: DETERMINISTIC SIGNAL

TF formulation. We shall now derive a TF formulation of the optimal test statistic (1) for detecting a deterministic signal. Applying (11) to (1), we obtain

$$\Lambda_o(r) = \mathrm{Re}\langle \mathbf{R}_n^{-1} r, s\rangle = \big\langle L_{\mathbf{R}_n^{-1}}, \mathrm{Re}\,\{W_{s,r}\}\big\rangle.$$

Assuming an underspread noise process, we can use (14),

$$L_{\mathbf{R}_n^{-1}}(t,f) \approx \frac{1}{L_{\mathbf{R}_n}(t,f)} = \frac{1}{\overline{W}_n(t,f)}\,, \quad (t,f) \in \mathcal{G}_n\,,$$

where $\mathcal{G}_n$, the *observation TF region* (corresponding to the observation space $\mathcal{S}_n$ [16]), is the effective TF support of the noise WVS $\overline{W}_n(t,f)$. Furthermore, since both $s(t)$ and $r(t)$ are in $\mathcal{S}_n$, $W_{s,r}(t,f)$ will be effectively zero outside $\mathcal{G}_n$, and we finally obtain the following approximate TF formulation of the optimal test statistic (1),

$$\Lambda_o(r) \approx \iint_{\mathcal{G}_n} \frac{\mathrm{Re}\,\{W_{s,r}(t,f)\}}{\overline{W}_n(t,f)}\, dt\, df. \qquad (15)$$

The deflection (2) can similarly be approximated as

$$d_o^2 \approx 2\iint_{\mathcal{G}_n} \frac{W_s(t,f)}{\overline{W}_n(t,f)}\, dt\, df = 2\iint_{\mathcal{G}_n} \mathrm{SNR}(t,f)\, dt\, df\,,$$

where $W_s(t,f) = W_{s,s}(t,f)$ is the auto Wigner distribution [13] of $s(t)$ and $\mathrm{SNR}(t,f) = W_s(t,f)/\overline{W}_n(t,f)$ is a "TF dependent signal to noise ratio." Thus, we obtained an intuitively pleasing interpretation of the deflection d_o^2 as average TF SNR. Note that the above TF expressions are completely analogous to the frequency-domain expressions (7) valid in the stationary case; indeed, they reduce to (7) for stationary noise.

TF design. The TF formulation (15) suggests a *TF design* of a test statistic for detecting a deterministic signal. The TF designed test statistic is defined as

$$\tilde{\Lambda}_o(r) \triangleq \iint_{\mathcal{G}_n} \frac{\mathrm{Re}\,\{W_{s,r}(t,f)\}}{\overline{W}_n(t,f)}\, dt\, df. \qquad (16)$$

For underspread noise where (15) is a good approximation, there is $\tilde{\Lambda}_o(r) \approx \Lambda_o(r)$ so that the TF designed detector will perform nearly as well as the optimal detector. The deflection of the TF designed detector is easily shown to be the average TF SNR,

$$\tilde{d}_o^2 = 2\iint_{\mathcal{G}_n} \frac{W_s(t,f)}{\overline{W}_n(t,f)}\, dt\, df = 2\iint_{\mathcal{G}_n} \mathrm{SNR}(t,f)\, dt\, df\,. \quad (17)$$

The TF designed test statistic can equivalently be expressed as the linear functional $\tilde{\Lambda}_o(r) = \mathrm{Re}\langle \widetilde{\mathbf{H}}_o r, s\rangle$, where $\widetilde{\mathbf{H}}_o$ is defined via its WS as $L_{\widetilde{\mathbf{H}}_o}(t,f) \triangleq 1/\overline{W}_n(t,f)$, $(t,f) \in \mathcal{G}_n$. The kernel of $\widetilde{\mathbf{H}}_o$ can be obtained using (10). This can be viewed as an (approximate) TF implementation of the inversion of $\mathbf{R}_n$, which is computationally attractive as the WS allows an efficient implementation using the FFT. The

TF designed detector has the further advantage that the required *a priori* knowledge (the noise WVS $\overline{W}_n(t,f)$) is specified in the intuitively more accessible TF domain.

Optimal signal design. Returning to the optimal test statistic (1), we define the *optimal signal* $s_{\text{opt}}(t)$ as the (normalized) signal $s(t)$ maximizing the deflection,

$$s_{\text{opt}}(t) \;=\; \arg\max_{\|s\|=1} d_o^2 \;=\; \arg\max_{\|s\|=1} \langle \mathbf{R}_n^{-1} s, s\rangle.$$

Assuming that $\mathbf{R}_n$ has finite rank, $s_{\text{opt}}(t)$ can be shown to be the eigenfunction of $\mathbf{R}_n$ corresponding to the smallest eigenvalue of $\mathbf{R}_n$. With (11), we have equivalently

$$s_{\text{opt}}(t) = \arg\max_{\|s\|=1} \langle L_{\mathbf{R}_n^{-1}}, W_s\rangle. \qquad (18)$$

This can be interpreted as a *TF signal synthesis* problem [17, 18]. TF signal synthesis is the calculation of the (normalized) signal $s(t)$ whose Wigner distribution, $W_s(t,f)$, is closest to a given "TF model function" $M(t,f)$, i.e., $s_{\text{opt}}(t) = \arg\min_{\|s\|=1} \|M - W_s\|$ or, equivalently,

$$s_{\text{opt}}(t) \;=\; \arg\max_{\|s\|=1} \langle M, W_s\rangle\,.$$

This is recognized as our signal design problem (18) with TF model $M(t,f) = L_{\mathbf{R}_n^{-1}}(t,f)$. For an underspread noise process, there is $L_{\mathbf{R}_n^{-1}}(t,f) \approx 1/\overline{W}_n(t,f)$ which shows that the optimum signal will occupy those TF regions where the noise WVS assumes small values.

In a similar manner, we can maximize the deflection (17) of the TF designed detector (16),

$$\tilde{s}_{\text{opt}}(t) = \arg\max_{\|s\|=1} \iint_{\mathcal{G}_n} \text{SNR}(t,f)\, dt\, df$$

where $\text{SNR}(t,f) = W_s(t,f)/\overline{W}_n(t,f)$. This corresponds to TF signal synthesis with $M(t,f) = 1/\overline{W}_n(t,f)$.

4 CASE 2: RANDOM SIGNAL

TF formulation. Assuming that the signal process $s(t)$ and the noise process $n(t)$ are jointly underspread, and reasoning as in the previous section, the WSs of $\mathbf{H}_l$ in (4) and $\mathbf{H}_d$ in (5) can be approximated as $L_{\mathbf{H}_l}(t,f) \approx \frac{\overline{W}_s(t,f)}{\overline{W}_n(t,f)\,[\overline{W}_s(t,f)+\overline{W}_n(t,f)]}$ and $L_{\mathbf{H}_d}(t,f) \approx \frac{\overline{W}_s(t,f)}{\overline{W}_n^2(t,f)}$ for $(t,f) \in \mathcal{G}_s$; here, $\mathcal{G}_s \subseteq \mathcal{G}_n$ denotes the effective support of $\overline{W}_s(t,f)$. This results in the following approximate TF formulations of the test statistics $\Lambda_l(r)$ in (3) and $\Lambda_d(r)$ in (5),

$$\Lambda_l(r) \approx \iint_{\mathcal{G}_s} \frac{\overline{W}_s(t,f)}{\overline{W}_n(t,f)\,[\overline{W}_s(t,f)+\overline{W}_n(t,f)]}\, W_r(t,f)\, dt\, df, \quad (19)$$

$$\Lambda_d(r) \approx \iint_{\mathcal{G}_s} \frac{\overline{W}_s(t,f)}{\overline{W}_n^2(t,f)} W_r(t,f)\, dt\, df. \qquad (20)$$

Furthermore, the (maximum) deflection achieved by the deflection optimal detector $\Lambda_d(r)$ can be reformulated as

$$d_d^2 = \int_t \int_f L_{\mathbf{\Gamma}^2}(t,f)\, dt\, df \approx \iint_{\mathcal{G}_s} \text{SNR}^2(t,f)\, dt\, df\,,$$

with the TF dependent signal to noise ratio defined for $(t,f) \in \mathcal{G}_s$ as $\text{SNR}(t,f) = \overline{W}_s(t,f)/\overline{W}_n(t,f)$. These expressions generalize the frequency-domain expressions (8) and (9) valid in the stationary case; they reduce to (8) and (9) for stationary signal and noise processes.

TF design. The TF formulations (19) and (20) suggest a TF design resulting in the quadratic test statistics

$$\tilde{\Lambda}_l(r) \triangleq \iint_{\mathcal{G}_s} \frac{\overline{W}_s(t,f)}{\overline{W}_n(t,f)\,[\overline{W}_s(t,f)+\overline{W}_n(t,f)]}\, W_r(t,f)\, dt\, df, \quad (21)$$

$$\tilde{\Lambda}_d(r) \triangleq \iint_{\mathcal{G}_s} \frac{\overline{W}_s(t,f)}{\overline{W}_n^2(t,f)} W_r(t,f)\, dt\, df. \qquad (22)$$

For jointly underspread signal and noise, the TF designed test statistics $\tilde{\Lambda}_l(r)$ and $\tilde{\Lambda}_d(r)$ can be expected to perform nearly as well as the optimal test statistics $\Lambda_l(r)$ and $\Lambda_d(r)$, respectively. The deflection achieved by $\tilde{\Lambda}_d(r)$ is given by

$$\tilde{d}_d^2 = \iint_{\mathcal{G}_s} \text{SNR}^2(t,f)\, dt\, df = \|\text{SNR}\|^2.$$

The above TF designed test statistics can be equivalently implemented as quadratic forms, $\tilde{\Lambda}_l(r) = \langle \widetilde{\mathbf{H}}_l r, r\rangle$ and $\tilde{\Lambda}_d(r) = \langle \widetilde{\mathbf{H}}_d r, r\rangle$, where $\widetilde{\mathbf{H}}_l$ and $\widetilde{\mathbf{H}}_d$ are defined via their WSs as $L_{\mathbf{H}_l}(t,f) \triangleq \frac{\overline{W}_s(t,f)}{\overline{W}_n(t,f)\,[\overline{W}_s(t,f)+\overline{W}_n(t,f)]}$ and $L_{\mathbf{H}_d}(t,f) \triangleq \frac{\overline{W}_s(t,f)}{\overline{W}_n^2(t,f)}$ for $(t,f) \in \mathcal{G}_s$.

The TF designed detectors have the advantage that the *a priori* knowledge required is specified in the TF domain. Furthermore, the operator inversions of $\mathbf{R}_n$ and $\mathbf{R}_s + \mathbf{R}_n$ in (4) and (5) are replaced by computationally less expensive scalar inversions plus WS transforms.

Simulation results. *Fig. 1* compares the performance of the optimal likelihood ratio detector (test statistic $\Lambda_l(r)$) with that of the corresponding TF designed detector (test statistic $\tilde{\Lambda}_l(r)$). The nonstationary processes $s(t)$ and $n(t)$ were generated using the TF synthesis method in [19]. The performance results in parts (e)-(h) were obtained by Monte Carlo simulation. It is seen that the TF designed detector closely approximates the optimal detector.

5 EXTENDED TIME-FREQUENCY DESIGN

We now consider a nonstationary random signal $s(t)$ corrupted by *stationary white* noise with known intensity (power spectral density) η. The statistics of the signal process $s(t)$ are assumed to be unknown except for the support $\mathcal{A}_s$ of the EAF of $s(t)$ (see (13)). This reduced *a priori* knowledge suffices to calculate a minimum-variance unbiased estimate of the signal WVS $\overline{W}_s(t,f)$ given by [20]

$$\widehat{\overline{W}}_s(t,f) = \langle \mathbf{T}_{t,f} r, r\rangle - \eta \qquad \text{with } \mathbf{T}_{t,f} = \mathbf{S}_{t,f} \mathbf{T} \mathbf{S}_{t,f}^{-1}.$$

Here, $\mathbf{S}_{t,f}$ is the TF shift operator, $(\mathbf{S}_{t,f}x)(t') = x(t' - t)\, e^{j2\pi f t'}$, and the operator $\mathbf{T}$ is defined via its spreading function (see (12)) such that $S_{\mathbf{T}}(\tau,\nu) = I_s(\tau,\nu)$, where $I_s(\tau,\nu)$ is the indicator function of the EAF support $\mathcal{A}_s$ (i.e., $I_s(\tau,\nu)$ is 1 for $(\tau,\nu) \in \mathcal{A}_s$ and 0 elsewhere). This WVS estimator can be shown [20] to be unbiased under hypothesis H_1 (i.e., when the signal $s(t)$ is actually present). It can also be shown [20] that the variance of $\widehat{\overline{W}}_s(t,f)$ will be reasonably small if $s(t)$ is underspread.

Substituting the WVS estimate $\widehat{\overline{W}}_s(t,f)$ for $\overline{W}_s(t,f)$ and using $\overline{W}_n(t,f) = \eta$ in the TF designed test statistics $\tilde{\Lambda}_l(r)$ and $\tilde{\Lambda}_d(r)$ (see (21), (22)) yields the test statistics

$$\Lambda_l'(r) = \frac{1}{\eta} \int_t \int_f \frac{\widehat{\overline{W}}_s(t,f)}{\widehat{\overline{W}}_s(t,f) + \eta}\, W_r(t,f)\, dt\, df\,,$$

$$\Lambda_d'(r) = \frac{1}{\eta^2} \int_t \int_f \widehat{\overline{W}}_s(t,f)\, W_r(t,f)\, dt\, df\,.$$

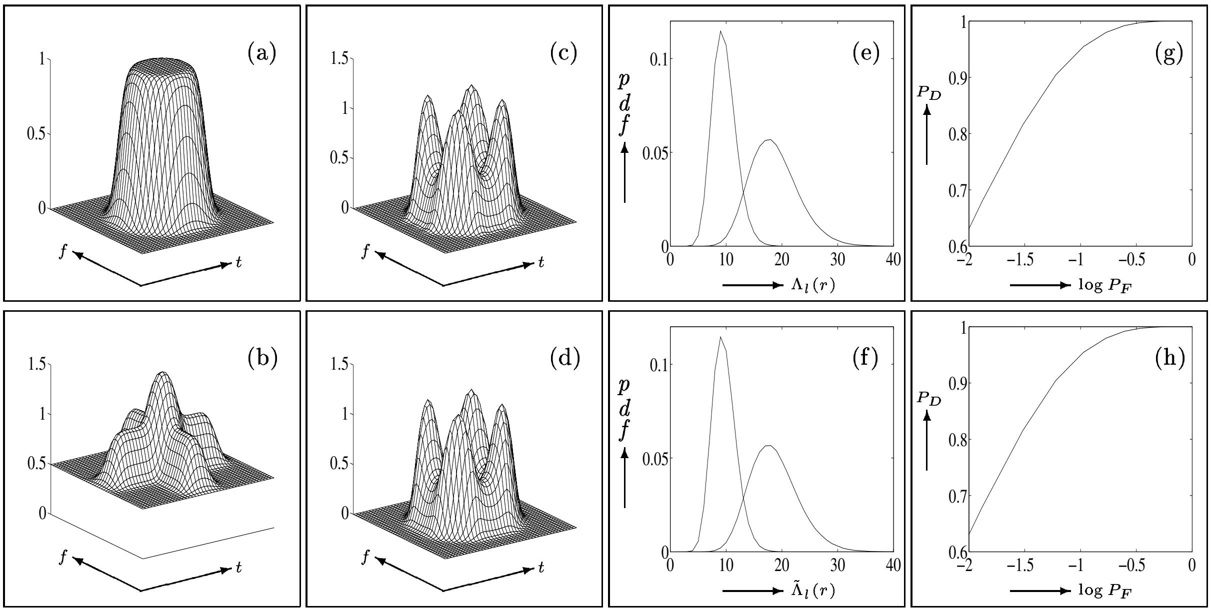

Figure 1. *Comparison of optimal and TF designed detectors: (a) WVS of underspread signal $s(t)$, (b) WVS of underspread noise $n(t)$, (c) WS of optimal operator $\mathbf{H}_l$, (d) WS of TF designed operator $\widetilde{\mathbf{H}}_l$, (e) conditional probability density functions (pdf's) of optimal test statistic $\Lambda_l(r)$ under either hypothesis, (f) conditional pdf's of TF designed test statistic $\tilde{\Lambda}_l(r)$, (g) receiver operator characteristics (ROC) [3] of optimal test statistic $\Lambda_l(r)$, and (h) ROC of TF designed test statistic $\tilde{\Lambda}_l(r)$.*

For $s(t)$ underspread, these test statistics can be expected to perform reasonably well. Note that $\Lambda'_d(r)$ allows an intuitive "estimator-correlator" interpretation: an estimate of the signal WVS is computed which is then correlated with the Wigner distribution of the observation $r(t)$.

6 CONCLUSIONS

We presented a framework for the time-frequency formulation, interpretation, and design of optimal detectors for deterministic and random signals. This framework is based on the Weyl symbol and Wigner-Ville spectrum, and is valid for *underspread*, nonstationary processes. We also extended the time-frequency detector design to include an estimation of the signal's Wigner-Ville spectrum.

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