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TIME-FREQUENCY COHERENCE ANALYSIS OF NONSTATIONARY RANDOM PROCESSES*

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ABSTRACT

The coherence function is extended to nonstationary random processes through introduction and investigation of a *coherence operator* and *time-frequency (TF) coherence functions*. For *underspread* nonstationary processes, it is shown that TF coherence functions are a meaningful tool for nonstationary coherence analysis and that they provide approximate TF formulations of the coherence operator.

1. INTRODUCTION

Consider two jointly stationary, zero-mean, real or circular complex random processes $x(t)$ and $y(t)$ with power spectral densities $P_x(f)$ and $P_y(f)$ and cross power spectral density $P_{x,y}(f)$. The *coherence function* [1–3]

$$\gamma_{x,y}(f) \triangleq \frac{P_{x,y}(f)}{\sqrt{P_x(f)\,P_y(f)}}$$

is a practically useful, normalized measure of the cross-correlation of spectral components of $x(t)$ and $y(t)$. It satisfies

$$|\gamma_{x,y}(f)|^2 \le 1\,, \tag{1}$$

with $|\gamma_{x,y}(f)|^2 \equiv 0$ iff $x(t)$ and $y(t)$ are uncorrelated processes ($P_{x,y}(f)=0$) and

$$|\gamma_{x,y}(f)|^2 \equiv 1 \tag{2}$$

iff $x(t)$ and $y(t)$ are related by an invertible linear time-invariant system, $y(t)=(k*x)(t)$. Furthermore, $|\gamma_{x,y}(f)|^2$ is invariant to invertible linear process transformations, i.e., for $a(t)=(h_1*x)(t)$ and $b(t)=(h_2*y)(t)$ we have

$$|\gamma_{a,b}(f)|^2 = |\gamma_{x,y}(f)|^2\,. \tag{3}$$

This paper extends the coherence function to nonstationary processes. Section 2 introduces and studies a *coherence operator* of nonstationary processes. Section 3 reviews some time-frequency (TF) fundamentals. Section 4 shows that for *underspread* nonstationary processes, the TF coherence function introduced in [4] is an approximate TF formulation of the coherence operator that approximately satisfies several desirable properties. Section 5 introduces a class of TF shift covariant TF coherence functions. Simulation results are presented in Section 6. We note that proofs are omitted due to lack of space; most proofs can be found in [5].

2. THE COHERENCE OPERATOR

Let us consider two *nonstationary*, zero-mean, real or circular complex random processes $x(t)$ and $y(t)$ with autocorrelation operators $\mathbf{R}_x$, $\mathbf{R}_y$ and cross-correlation operator[1] $\mathbf{R}_{x,y}$. The coherence function is no longer defined; however, by analogy to the coherence matrix [7], we define the *coherence operator* of $x(t)$ and $y(t)$ as

$$\mathbf{\Gamma}_{x,y} \triangleq \mathbf{R}_x^{-1/2}\mathbf{R}_{x,y}\mathbf{R}_y^{-1/2},$$

where, e.g., $\mathbf{R}_x^{-1/2}$ denotes the inverse of the positive semi-definite square-root $\mathbf{R}_x^{1/2}$ of $\mathbf{R}_x$ [6]. Equivalently,

$$\mathbf{\Gamma}_{x,y} = \mathbf{R}_{\tilde{x},\tilde{y}}\,,$$

where $\tilde{x}(t) = \big(\mathbf{R}_x^{-1/2}x\big)(t)$ and $\tilde{y}(t) = \big(\mathbf{R}_y^{-1/2}y\big)(t)$ are stationary and white with correlation $\mathbf{R}_{\tilde{x}} = \mathbf{R}_{\tilde{y}} = \mathbf{I}$.

If $x(t)$ and $y(t)$ are jointly stationary, the kernel of $\mathbf{\Gamma}_{x,y}$ is given by $(\mathbf{\Gamma}_{x,y})(t_1,t_2) = \tilde{\gamma}_{x,y}(t_1-t_2)$ with $\tilde{\gamma}_{x,y}(\tau) = \int_{-\infty}^{\infty}\gamma_{x,y}(f)\,e^{j2\pi f\tau}df$. In this sense, $\mathbf{\Gamma}_{x,y}$ is consistent with the conventional coherence function $\gamma_{x,y}(f)$.

Bounds. The coherence operator $\mathbf{\Gamma}_{x,y}$ satisfies bounds that are analogous to (1). Specifically, the singular values [6] $\sigma_k \ge 0$ of $\mathbf{\Gamma}_{x,y}$ are bounded as

$$\sigma_k \le 1\,.$$

The operator norm [6] $\|\mathbf{\Gamma}_{x,y}\|_{\mathrm{O}} \triangleq \sup_{\|g\|_2=1}\|\mathbf{\Gamma}_{x,y}g\|_2$ (with $\|g\|_2 = \big[\int_{-\infty}^{\infty}|g(t)|^2dt\big]^{1/2}$) is similarly bounded as

$$\|\mathbf{\Gamma}_{x,y}\|_{\mathrm{O}} \le 1\,.$$

Finally, we have the following bounds on the (non-negative) quadratic forms induced by the positive semi-definite [6] "squared" coherence operators $\mathbf{\Gamma}_{x,y}\mathbf{\Gamma}_{x,y}^+$ or $\mathbf{\Gamma}_{x,y}^+\mathbf{\Gamma}_{x,y}$ (with $\mathbf{\Gamma}_{x,y}^+$ the adjoint [6] of $\mathbf{\Gamma}_{x,y}$): for any $g(t)$ with $\|g\|_2=1$,

$$\langle\mathbf{\Gamma}_{x,y}\mathbf{\Gamma}_{x,y}^+g,g\rangle \le 1\,, \qquad \langle\mathbf{\Gamma}_{x,y}^+\mathbf{\Gamma}_{x,y}g,g\rangle \le 1\,,$$

with the inner product defined as $\langle x,y\rangle \triangleq \int_{-\infty}^{\infty}x(t)y^*(t)\,dt$. Note that $\mathbf{\Gamma}_{x,y} = \mathbf{0}$ iff $x(t)$ and $y(t)$ are uncorrelated.

Completely coherent processes. The "squared" coherence operators equal the identity operator,

$$\mathbf{\Gamma}_{x,y}\mathbf{\Gamma}_{x,y}^+ = \mathbf{\Gamma}_{x,y}^+\mathbf{\Gamma}_{x,y} = \mathbf{I}$$

(equivalently, $\mathbf{\Gamma}_{x,y}$ is a *unitary* operator [6]), iff $y(t)=(\mathbf{K}x)(t)$ with some invertible linear (generally time-varying) system $\mathbf{K}$. This extends property (2) to the nonstationary case.

Linearly distorted processes. An extension of (3) is possible only under rather restrictive assumptions. Let

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[1] $\mathbf{R}_{x,y}$ is the linear operator [6] with kernel $r_{x,y}(t_1,t_2) = \mathrm{E}\{x(t_1)\,y^*(t_2)\}$; furthermore, $\mathbf{R}_x = \mathbf{R}_{x,x}$.

$a(t) = (\mathbf{H}_1 x)(t)$ and $b(t) = (\mathbf{H}_2 y)(t)$ with $\mathbf{H}_1$ and $\mathbf{H}_2$ invertible. Then $\mathbf{\Gamma}_{a,b}\mathbf{\Gamma}_{a,b}^{+} = \mathbf{\Gamma}_{x,y}\mathbf{\Gamma}_{x,y}^{+}$ if $\mathbf{H}_1$ is positive definite and commutes with $\mathbf{R}_x$. Similarly, $\mathbf{\Gamma}_{a,b}^{+}\mathbf{\Gamma}_{a,b} = \mathbf{\Gamma}_{x,y}^{+}\mathbf{\Gamma}_{x,y}$ if $\mathbf{H}_2$ is positive definite and commutes with $\mathbf{R}_y$.

3. TIME-FREQUENCY FUNDAMENTALS

Next, we briefly review some TF representations and concepts that will be used in subsequent sections.

• The *Weyl symbol* (WS) [8–11] of a linear operator (linear time-varying system) $\mathbf{H}$ with kernel (impulse response) $h(t_1, t_2)$ is defined as

$$L_{\mathbf{H}}(t,f) \triangleq \int_{-\infty}^{\infty} h\Big(t+\frac{\tau}{2}, t-\frac{\tau}{2}\Big)\, e^{-j2\pi f\tau}\, d\tau \,.$$

For an *underspread* system $\mathbf{H}$ (see below), $L_{\mathbf{H}}(t,f)$ can be viewed as a time-varying transfer function [10, 11].

• The *spreading function* (SF) [8,10,11] of a linear operator (linear time-varying system) $\mathbf{H}$,

$$S_{\mathbf{H}}(\tau,\nu) \triangleq \int_{-\infty}^{\infty} h\Big(t+\frac{\tau}{2}, t-\frac{\tau}{2}\Big)\, e^{-j2\pi\nu t}\, dt \,,$$

describes the distribution of time shifts by τ and frequency shifts by ν effected by $\mathbf{H}$.

• The *Wigner-Ville spectrum* (WVS) [12–15] of nonstationary random processes $x(t)$, $y(t)$ is defined as

$$\overline{W}_{x,y}(t,f) \triangleq L_{\mathbf{R}_{x,y}}(t,f)\,, \quad \overline{W}_{x}(t,f) \triangleq \overline{W}_{x,x}(t,f) \in \mathbb{R}\,.$$

For *jointly underspread* processes $x(t)$, $y(t)$ (see below), it can be interpreted as a time-varying (cross) power spectrum.

• Another time-varying power spectrum is the *physical spectrum* [12–16]

$$\bar{S}_{x,y}(t,f) \triangleq \overline{W}_{x,y}(t,f) ** W_g(-t,-f)$$
$$\bar{S}_{x}(t,f) \triangleq \bar{S}_{x,x}(t,f) \geq 0\,,$$

where $**$ denotes 2-D convolution, $g(t)$ is an analysis window (normalized such that $\|g\|_2 = 1$), and $W_g(t,f)$ is the Wigner distribution [12, 17] of $g(t)$.

• The *expected ambiguity function* [14, 18]

$$\bar{A}_{x,y}(\tau,\nu) \triangleq S_{\mathbf{R}_{x,y}}(\tau,\nu)\,, \quad \bar{A}_{x}(\tau,\nu) \triangleq \bar{A}_{x,x}(\tau,\nu)$$

describes the statistical correlation of process components separated in time by τ and in frequency by ν.

• A system/operator $\mathbf{H}$ is *underspread* if its SF $S_{\mathbf{H}}(\tau,\nu)$ is supported within a rectangular region $\mathcal{G} = [-\tau_{\mathcal{G}}, \tau_{\mathcal{G}}] \times [-\nu_{\mathcal{G}}, \nu_{\mathcal{G}}]$ of area $\sigma_{\mathcal{G}} = 4\tau_{\mathcal{G}}\nu_{\mathcal{G}} \ll 1$ [5,10,11]. This means that $\mathbf{H}$ introduces only small TF shifts. Similarly, a process $x(t)$ is underspread if its expected ambiguity function $\bar{A}_x(\tau,\nu)$ is supported within a rectangular region $\mathcal{G}$ of area $\sigma_{\mathcal{G}} \ll 1$ [14, 18]. This means that $x(t)$ features only limited TF correlations. Two processes $x(t), y(t)$ are *jointly underspread* if $\bar{A}_x(\tau,\nu)$, $\bar{A}_y(\tau,\nu)$, and $\bar{A}_{x,y}(\tau,\nu)$ are supported within the *same* rectangular region of area $\sigma_{\mathcal{G}} \ll 1$.

4. A TIME-FREQUENCY COHERENCE FUNCTION

We are now ready to study a simple and intuitively appealing TF formulation of the coherence operator $\mathbf{\Gamma}_{x,y}$ that avoids operator inversions. A *TF coherence function* based on the WVS was defined in [4] as

$$\Gamma_{x,y}(t,f) \triangleq \frac{\overline{W}_{x,y}(t,f)}{\sqrt{\overline{W}_x(t,f)\,\overline{W}_y(t,f)}}\,, \qquad (t,f) \in \mathcal{R}\,, \qquad (4)$$

where $\mathcal{R}$ is the TF region on which $\overline{W}_x(t,f) > 0$ and $\overline{W}_y(t,f) > 0$. $\Gamma_{x,y}(t,f)$ is a complex-valued function that is covariant to TF shifts (see Section 5) as well as to TF scalings and other metaplectic transformations of $x(t), y(t)$. For $x(t), y(t)$ uncorrelated, there is $\Gamma_{x,y}(t,f) \equiv 0$ on $\mathcal{R}$.

$\Gamma_{x,y}(t,f)$ as approximate TF formulation of $\mathbf{\Gamma}_{x,y}$. We now show that for $x(t), y(t)$ jointly underspread, $\Gamma_{x,y}(t,f)$ approximates the WS of the coherence operator $\mathbf{\Gamma}_{x,y}$. We start by noting that $\mathbf{\Gamma}_{x,y}$ can be alternatively defined by

$$\mathbf{H}_x \mathbf{\Gamma}_{x,y} \mathbf{H}_y = \mathbf{R}_{x,y}\,, \qquad (5)$$

with $\mathbf{H}_x = \mathbf{R}_x^{1/2}$ and $\mathbf{H}_y = \mathbf{R}_y^{1/2}$. Our central assumption will be that $S_{\mathbf{H}_x}(\tau,\nu)$, $S_{\mathbf{H}_y}(\tau,\nu)$, and $\bar{A}_{x,y}(\tau,\nu)$ are supported within the same rectangular region $\mathcal{G} = [-\tau_{\mathcal{G}}, \tau_{\mathcal{G}}] \times [-\nu_{\mathcal{G}}, \nu_{\mathcal{G}}]$ of area $\sigma_{\mathcal{G}} = 4\tau_{\mathcal{G}}\nu_{\mathcal{G}}$.

We can split the coherence operator $\mathbf{\Gamma}_{x,y}$ into a part $\mathbf{\Gamma}_{x,y}^{\mathcal{G}}$ whose SF is supported within $\mathcal{G}$ and a part $\mathbf{\Gamma}_{x,y}^{\overline{\mathcal{G}}}$ whose SF is supported outside $\mathcal{G}$. This is motivated by the desire of approximating $\mathbf{\Gamma}_{x,y}$ by $\mathbf{\Gamma}_{x,y}^{\mathcal{G}}$ in the sense that replacing $\mathbf{\Gamma}_{x,y}$ by $\mathbf{\Gamma}_{x,y}^{\mathcal{G}}$ does not greatly affect the validity of (5):

$$\mathbf{H}_x \mathbf{\Gamma}_{x,y} \mathbf{H}_y = \mathbf{R}_{x,y} \quad \Longrightarrow \quad \mathbf{H}_x \mathbf{\Gamma}_{x,y}^{\mathcal{G}} \mathbf{H}_y \approx \mathbf{R}_{x,y}\,. \qquad (6)$$

Indeed, we have the following result.

Theorem 1 [5]. *Under the assumption stated above, the difference $\mathbf{H}_x \mathbf{\Gamma}_{x,y}^{\mathcal{G}} \mathbf{H}_y - \mathbf{R}_{x,y}$ is bounded as*[2]

$$\frac{\|\mathbf{H}_x \mathbf{\Gamma}_{x,y}^{\mathcal{G}} \mathbf{H}_y - \mathbf{R}_{x,y}\|_2}{\|\mathbf{H}_x\|_2\, \|\mathbf{\Gamma}_{x,y}\|_2\, \|\mathbf{H}_y\|_2} \leq 3\sqrt{\sigma_{\mathcal{G}}}\,.$$

Hence, if $\sigma_{\mathcal{G}} \ll 1$, i.e., if $x(t)$ and $y(t)$ are jointly underspread, the approximation in (6) is indeed valid.

We now pass to the TF domain using the WS.

Theorem 2 [5]. *Under the assumption stated above, the difference $\Delta_1(t,f) \triangleq L_{\mathbf{H}_x}(t,f) L_{\mathbf{\Gamma}_{x,y}^{\mathcal{G}}}(t,f)\, L_{\mathbf{H}_y}(t,f) - \overline{W}_{x,y}(t,f)$ is bounded as*[3]

$$\frac{|\Delta_1(t,f)|}{\|S_{\mathbf{H}_x}\|_1\, \|S_{\mathbf{\Gamma}_{x,y}}\|_\infty\, \|S_{\mathbf{H}_y}\|_1} \leq \frac{3\pi}{2}\,\sigma_{\mathcal{G}}^2 + 9\sigma_{\mathcal{G}}\,.$$

Hence, for $\sigma_{\mathcal{G}} \ll 1$ one has

$$L_{\mathbf{H}_x}(t,f)\, L_{\mathbf{\Gamma}_{x,y}^{\mathcal{G}}}(t,f)\, L_{\mathbf{H}_y}(t,f) \approx \overline{W}_{x,y}(t,f)\,. \qquad (7)$$

We now insert the approximations $L_{\mathbf{H}_x}(t,f) \approx \sqrt{\overline{W}_x(t,f)}$ and $L_{\mathbf{H}_y}(t,f) \approx \sqrt{\overline{W}_y(t,f)}$ valid for underspread $x(t)$ and for underspread $y(t)$ [5] and divide by $\sqrt{\overline{W}_x(t,f)\,\overline{W}_y(t,f)}$ on $\mathcal{R}$. Equation (7) thus becomes

$$L_{\mathbf{\Gamma}_{x,y}^{\mathcal{G}}}(t,f) \approx \Gamma_{x,y}(t,f)\,, \qquad (t,f) \in \mathcal{R}\,, \qquad (8)$$

where $\Gamma_{x,y}(t,f)$ is the TF coherence function in (4). Furthermore, it can be shown [5] that $L_{\mathbf{\Gamma}_{x,y}^{\mathcal{G}}}(t,f)$ equals $L_{\mathbf{\Gamma}_{x,y}}(t,f)$

[2] Here, $\|\mathbf{H}\|_2 \triangleq \big[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h(t_1,t_2)|^2\, dt_1 dt_2\big]^{1/2}$.

[3] We note that $\|S_{\mathbf{H}}\|_1 \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |S_{\mathbf{H}}(\tau,\nu)|\, d\tau\, d\nu$ and $\|S_{\mathbf{H}}\|_\infty \triangleq \sup_{\tau,\nu} |S_{\mathbf{H}}(\tau,\nu)|$.

convolved by a function $\psi(t,f)$ whose 2-D Fourier transform is 1 in $\mathcal{G}$ and 0 outside $\mathcal{G}$. For $\sigma_{\mathcal{G}}$ small, $\psi(t,f)$ is a smooth function and thus $L_{\mathbf{\Gamma}^{\mathcal{G}}_{x,y}}(t,f)$ is a smoothed version of $L_{\mathbf{\Gamma}_{x,y}}(t,f)$. Hence, (8) states that *for jointly underspread* $x(t)$, $y(t)$, *the TF coherence function* $\Gamma_{x,y}(t,f)$ *is approximately equal to a smoothed version of the WS of the coherence operator* $\mathbf{\Gamma}_{x,y}$. In this sense, $\Gamma_{x,y}(t,f)$ provides an approximate TF formulation of the coherence operator $\mathbf{\Gamma}_{x,y}$.

Bounds. In Section 5, it will be shown that the alternative TF coherence function

$$\Gamma'_{x,y}(t,f) \triangleq \frac{\bar{S}_{x,y}(t,f)}{\sqrt{\bar{S}_x(t,f)\,\bar{S}_y(t,f)}} \tag{9}$$

satisfies the bound $|\Gamma'_{x,y}(t,f)|^2 \le 1$. For $x(t)$ and $y(t)$ jointly underspread, the WVS are approximately equal to the corresponding physical spectra [5,14,18], and thus $\Gamma_{x,y}(t,f) \approx \Gamma'_{x,y}(t,f)$ or, equivalently, $\bar{S}_x(t,f)\,\bar{S}_y(t,f)\,\big|\overline{W}_{x,y}(t,f)\big|^2 \approx \overline{W}_x(t,f)\,\overline{W}_y(t,f)\,\big|\bar{S}_{x,y}(t,f)\big|^2$. The last approximation is supported by the following result.

Theorem 3 [5]. *Let* $\bar{A}_x(\tau,\nu)$, $\bar{A}_y(\tau,\nu)$, *and* $\bar{A}_{x,y}(\tau,\nu)$ *be supported within the same rectangle* $\mathcal{G} = [-\tau_{\mathcal{G}},\tau_{\mathcal{G}}]\times[-\nu_{\mathcal{G}},\nu_{\mathcal{G}}]$. *Then, the difference* $\Delta_2(t,f) \triangleq \bar{S}_x(t,f)\,\bar{S}_y(t,f)\,\big|\overline{W}_{x,y}(t,f)\big|^2 - \overline{W}_x(t,f)\,\overline{W}_y(t,f)\,\big|\bar{S}_{x,y}(t,f)\big|^2$ *is bounded as*

$$\frac{\big|\Delta_2(t,f)\big|}{\|\bar{A}_x\|_1\,\|\bar{A}_y\|_1\,\|\bar{A}_{x,y}\|_1^2} \le 4\epsilon\,,$$

where $\epsilon \triangleq \max_{(\tau,\nu)\in\mathcal{G}}\big|1 - A_g(\tau,\nu)\big|$ *with* $A_g(\tau,\nu)$ *the ambiguity function [12, 17] of the analysis window* $g(t)$ *used in the physical spectrum.*

Since $A_g(\tau,\nu) \approx 1$ for small (τ,ν), ϵ will be small for small $\mathcal{G}$ and thus, still for small $\mathcal{G}$,

$$\big|\Gamma_{x,y}(t,f)\big|^2 \approx \big|\Gamma'_{x,y}(t,f)\big|^2. \tag{10}$$

With $|\Gamma'_{x,y}(t,f)|^2 \le 1$, (10) implies that for $x(t), y(t)$ jointly underspread, $|\Gamma_{x,y}(t,f)|^2$ is approximately bounded by 1.

However, for $x(t), y(t)$ *not* underspread, $|\Gamma_{x,y}(t,f)|^2$ may be *arbitrarily large*. Consider for example the two correlated random processes $x(t) = \beta\,u(t+t_0)\,e^{-j2\pi f_0 t}$ and $y(t) = \beta\,u(t-t_0)\,e^{j2\pi f_0 t}$ where $u(t) = e^{-\pi t^2/T^2}/\sqrt{2T}$, t_0 and f_0 are fixed, and β is random with $\mathrm{E}\{|\beta|^2\} = \gamma > 0$. One obtains

$$\overline{W}_{x,y}(t,f) = \gamma\, e^{-2\pi[t^2/T^2 + f^2T^2]}\, e^{j4\pi(t_0 f - f_0 t)}$$
$$\overline{W}_x(t,f) = \gamma\, e^{-2\pi[(t+t_0)^2/T^2 + (f+f_0)^2T^2]}$$
$$\overline{W}_y(t,f) = \gamma\, e^{-2\pi[(t-t_0)^2/T^2 + (f-f_0)^2T^2]}.$$

It is seen that $\overline{W}_x(t,f)$ and $\overline{W}_y(t,f)$ are localized about $(-t_0,-f_0)$ and (t_0,f_0), respectively. However, $\overline{W}_{x,y}(t,f)$ is localized (and oscillatory) about $(0,0)$, corresponding to a "statistical cross term" [14]. It follows that

$$|\Gamma_{x,y}(t,f)| = e^{2\pi[t_0^2/T^2 + f_0^2T^2]}\,,$$

which for increasing t_0, f_0 can become arbitrarily large. This refutes a previous incorrect claim that $|\Gamma_{x,y}(t,f)|$ is bounded by 1 [4]. Furthermore, we see that the large values of $|\Gamma_{x,y}(t,f)|$ are due to *TF correlations* [5,14,18], i.e., correlations between components of $x(t)$ and $y(t)$ located in different parts of the TF plane, which give rise to "statistical cross terms" in $\overline{W}_{x,y}(t,f)$ [14]. We note that for large t_0, f_0, the processes $x(t)$ and $y(t)$ are not jointly underspread.

Completely coherent processes. We next consider the case of linearly related processes $y(t) = (\mathbf{K}x)(t)$, where we would like to have

$$\big|\Gamma_{x,y}(t,f)\big|^2 \approx 1\,, \qquad (t,f)\in\mathcal{R} \tag{11}$$

or, equivalently, $\big|\overline{W}_{x,y}(t,f)\big|^2 \approx \overline{W}_x(t,f)\,\overline{W}_y(t,f)$.

Theorem 4 [5]. *Let* $S_{\mathbf{K}}(\tau,\nu)$ *and* $\bar{A}_x(\tau,\nu)$ *be supported within the same rectangle* $\mathcal{G} = [-\tau_{\mathcal{G}},\tau_{\mathcal{G}}]\times[-\nu_{\mathcal{G}},\nu_{\mathcal{G}}]$ *of area* $\sigma_{\mathcal{G}} = 4\tau_{\mathcal{G}}\nu_{\mathcal{G}}$. *Then, the difference* $\Delta_3(t,f) \triangleq \big|\overline{W}_{x,y}(t,f)\big|^2 - \overline{W}_x(t,f)\,\overline{W}_y(t,f)$ *is bounded as*

$$\frac{\big|\Delta_3(t,f)\big|}{\|\bar{A}_x\|_1^2\,\|S_{\mathbf{K}}\|_1^2} \le \frac{11\pi}{2}\,\sigma_{\mathcal{G}}\,.$$

Hence, for small $\sigma_{\mathcal{G}}$, $\big|\overline{W}_{x,y}(t,f)\big|^2 \approx \overline{W}_x(t,f)\,\overline{W}_y(t,f)$ and the approximation (11) is indeed valid. Small $\sigma_{\mathcal{G}}$ implies that $x(t)$ and $\mathbf{K}$ are jointly underspread; in this case, $y(t) = (\mathbf{K}x)(t)$ will be underspread as well. An example where $\mathbf{K}$ is *not* underspread and thus (11) is not valid was given further above. Indeed, the processes $x(t) = \beta\,u(t+t_0)\,e^{-j2\pi f_0 t}$ and $y(t) = \beta\,u(t-t_0)\,e^{j2\pi f_0 t}$ defined above are related as $y(t) = (\mathbf{K}x)(t)$, where $\mathbf{K}$ is a TF shift operator which for large t_0, f_0 is not underspread.

Linearly distorted processes. For $a(t) = (\mathbf{H}_1 x)(t)$ and $b(t) = (\mathbf{H}_2 y)(t)$, we would like to have the (approximate) invariance

$$\big|\Gamma_{a,b}(t,f)\big|^2 \approx \big|\Gamma_{x,y}(t,f)\big|^2\,, \qquad (t,f)\in\mathcal{R}\,, \tag{12}$$

which equivalently requires $|\overline{W}_{a,b}(t,f)|^2\,\overline{W}_x(t,f)\,\overline{W}_y(t,f) \approx |\overline{W}_{x,y}(t,f)|^2\,\overline{W}_a(t,f)\,\overline{W}_b(t,f)$.

Theorem 5 [5]. *Let* $S_{\mathbf{H}_1}(\tau,\nu)$, $S_{\mathbf{H}_2}(\tau,\nu)$, $\bar{A}_x(\tau,\nu)$, $\bar{A}_y(\tau,\nu)$, *and* $\bar{A}_{x,y}(\tau,\nu)$ *be supported within the same rectangle* $\mathcal{G} = [-\tau_{\mathcal{G}},\tau_{\mathcal{G}}]\times[-\nu_{\mathcal{G}},\nu_{\mathcal{G}}]$ *of area* $\sigma_{\mathcal{G}} = 4\tau_{\mathcal{G}}\nu_{\mathcal{G}}$. *Then, the difference* $\Delta_4(t,f) \triangleq |\overline{W}_{a,b}(t,f)|^2\,\overline{W}_x(t,f)\,\overline{W}_y(t,f) - |\overline{W}_{x,y}(t,f)|^2\,\overline{W}_a(t,f)\,\overline{W}_b(t,f)$ *is bounded as*

$$\frac{\big|\Delta_4(t,f)\big|}{\|\bar{A}_x\|_1\,\|\bar{A}_y\|_1\,\|\bar{A}_{x,y}\|_1^2\,\|S_{\mathbf{H}_1}\|_1^2\,\|S_{\mathbf{H}_2}\|_1^2} \le \frac{9\pi}{2}\,\sigma_{\mathcal{G}}\,.$$

Hence, for small $\sigma_{\mathcal{G}}$, (12) is indeed valid, which means that $\big|\Gamma_{x,y}(t,f)\big|^2$ is approximately invariant to linear process transformations. Small $\sigma_{\mathcal{G}}$ implies that the processes $x(t)$ and $y(t)$ and the operators $\mathbf{H}_1$ and $\mathbf{H}_2$ are all jointly underspread; this implies in turn that $a(t) = (\mathbf{H}_1 x)(t)$ and $b(t) = (\mathbf{H}_2 y)(t)$ are jointly underspread processes as well.

5. SHIFT-COVARIANT TIME-FREQUENCY COHERENCE FUNCTIONS

A generalization of $\Gamma_{x,y}(t,f)$ is given by

$$\Gamma^{(c)}_{x,y}(t,f) \triangleq \frac{P^{(c)}_{x,y}(t,f)}{\sqrt{P^{(c)}_x(t,f)\,P^{(c)}_y(t,f)}}\,, \qquad (t,f)\in\mathcal{R}\,,$$

where

$$P^{(c)}_{x,y}(t,f) \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r_{x,y}(t_1+t, t_2+t)\,c^*(t_1,t_2) \cdot e^{-j2\pi(t_1-t_2)f}\,dt_1 dt_2 \tag{13}$$

is a *TF shift covariant time-varying power spectrum* [5, 12–15] and $\mathcal{R}$ is the TF region on which $P_x^{(c)}(t,f) \triangleq P_{x,x}^{(c)}(t,f) > 0$ and $P_y^{(c)}(t,f) > 0$. We assume that the kernel function $c(t_1,t_2)$ in (13) satisfies $c^*(t_2,t_1) = c(t_1,t_2)$ so that $P_x^{(c)}(t,f)$ is real-valued. Two important special cases of $\Gamma_{x,y}^{(c)}(t,f)$ are $\Gamma_{x,y}(t,f)$ in (4) (obtained with $P_{x,y}^{(c)}(t,f) = \overline{W}_{x,y}(t,f)$ or $c(t_1,t_2) = \delta\big(\frac{t_1+t_2}{2}\big)$) and $\Gamma'_{x,y}(t,f)$ in (9) (obtained with $P_{x,y}^{(c)}(t,f) = \bar{S}_{x,y}(t,f)$ or $c(t_1,t_2) = g(t_1)g^*(t_2)$).

The TF coherence function $\Gamma_{x,y}^{(c)}(t,f)$ is a complex-valued function that is parameterized by $c(t_1,t_2)$ or, equivalently, by the (self-adjoint) linear operator $\mathbf{C}$ with kernel $c(t_1,t_2)$. For fixed $c(t_1,t_2)$, $\Gamma_{x,y}^{(c)}(t,f)$ is *TF shift covariant*, i.e.,

$$\Gamma_{\tilde{x},\tilde{y}}^{(c)}(t,f) = \Gamma_{x,y}^{(c)}(t-\tau, f-\nu)$$

with $\tilde{x}(t) = x(t-\tau)\, e^{j2\pi\nu t}$, $\tilde{y}(t) = y(t-\tau)\, e^{j2\pi\nu t}$. For $x(t), y(t)$ uncorrelated, there is $\Gamma_{x,y}^{(c)}(t,f) \equiv 0$ on $\mathcal{R}$.

Theorem 6 [5]. *There is*

$$\left|\Gamma_{x,y}^{(c)}(t,f)\right|^2 \leq 1\,, \qquad (t,f) \in \mathcal{R}$$

for all $x(t), y(t)$ and with nonempty $\mathcal{R}$ iff the operator $\mathbf{C}$ underlying $\Gamma_{x,y}^{(c)}(t,f)$ is positive semidefinite.[4]

Indeed, if $\mathbf{C}$ is positive semidefinite, then $P_{x,y}^{(c)}(t,f)$ is a *smoothed* version of $\overline{W}_{x,y}(t,f)$ [14]; this smoothing suppresses the statistical cross terms of $\overline{W}_{x,y}(t,f)$ present in the overspread case and thus allows $\Gamma_{x,y}^{(c)}(t,f)$ to be properly bounded even for overspread processes.

The operator $\mathbf{C}$ underlying $\Gamma'_{x,y}(t,f)$ in (9) is positive semidefinite, and thus $|\Gamma'_{x,y}(t,f)|^2 \leq 1$. On the other hand, the operator $\mathbf{C}$ underlying $\Gamma_{x,y}(t,f)$ is *not* positive semidefinite, and indeed we have observed in Section 4 that $|\Gamma_{x,y}(t,f)|^2$ can be arbitrarily large (however, we recall that it is approximately bounded by 1 in the underspread case).

6. SIMULATION RESULTS

Experiment 1. We analyze the coherence of the input $x(t)$ and the noise-contaminated output $y(t) = (\mathbf{K}x)(t) + n(t)$ of a time-varying linear system $\mathbf{K}$. The input $x(t)$ is stationary and white with correlation $\mathbf{R}_x = \mathbf{I}$ (corresponding to constant WVS $\overline{W}_x(t,f) \equiv 1$). The noise $n(t)$ is stationary and white with correlation $\mathbf{R}_n = \eta\,\mathbf{I}$ (corresponding to constant WVS $\overline{W}_n(t,f) \equiv \eta$) and uncorrelated with $x(t)$. The WS and SF of $\mathbf{K}$ are depicted in Figs. 1(a) and (b), respectively. The SF of $\mathbf{K}$ shows that $\mathbf{K}$ is underspread.

In the noise-free case ($\eta = 0$), $x(t)$ and $y(t)$ are completely coherent, i.e., $\mathbf{\Gamma}_{x,y}\mathbf{\Gamma}_{x,y}^{+} = \mathbf{\Gamma}_{x,y}^{+}\mathbf{\Gamma}_{x,y} = \mathbf{I}$. Since $x(t)$ and $\mathbf{K}$ are underspread, Theorem 4 applies and we can expect that $|\Gamma_{x,y}(t,f)|^2 \approx 1$. Indeed, we found that the maximum deviation of $|\Gamma_{x,y}(t,f)|^2$ from 1 was 0.028.

For $\eta > 0$, the noise causes a reduction of coherence that depends on the output SNR. Since the output SNR is TF-dependent (due to the TF weighting characteristic of $\mathbf{K}$ as shown in Fig. 1(a)), the coherence reduction is TF-dependent as well. This is clearly indicated by the WS of $\mathbf{\Gamma}_{x,y}^{\mathcal{G}}$ and the TF coherence function $\Gamma_{x,y}(t,f)$ shown in Figs. 1(c) and (d), respectively. Moreover, the similarity of these two results confirms the validity of the approximation (8).

[4]We recall that a positive semidefinite operator $\mathbf{C}$ is defined by the condition $\langle \mathbf{C}x, x\rangle \geq 0$ for all $x(t)$ [6]. For $\mathbf{C}$ positive semidefinite, there is $P_x^{(c)}(t,f) \geq 0$ for all (t,f) and for all $x(t)$.

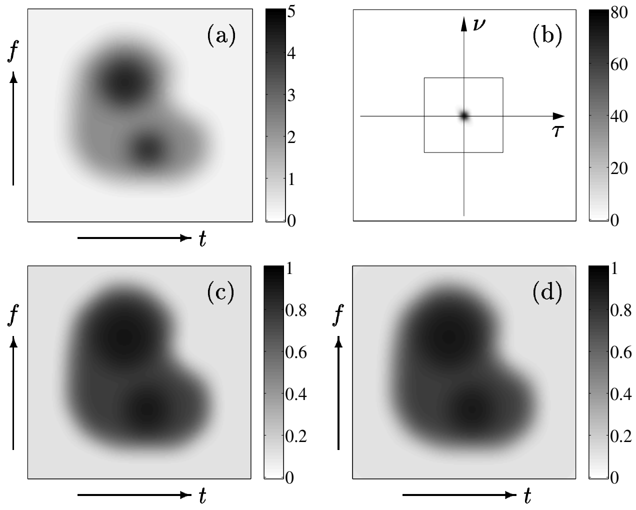

Figure 1: *Simulation results for Experiment 1: (a) WS of $\mathbf{K}$; (b) magnitude of SF of $\mathbf{K}$; (c) magnitude of WS of $\mathbf{\Gamma}_{x,y}^{\mathcal{G}}$; (d) magnitude of $\Gamma_{x,y}(t,f)$. The rectangle in (b) has area 1 and allows to assess the underspread property of $\mathbf{K}$. Time duration is 256 samples; normalized frequency ranges from $-1/4$ to $1/4$.*

Experiment 2. Again, $y(t) = (\mathbf{K}x)(t) + n(t)$ with $x(t)$ and $\mathbf{K}$ as in the previous example. However, $n(t)$ now is nonstationary narrowband noise with WVS as shown in Fig. 2(a). From the expected ambiguity function of $n(t)$ shown in Fig. 2(b), it is seen that $n(t)$ is reasonably underspread. The Weyl symbol of $\mathbf{\Gamma}_{x,y}^{\mathcal{G}}$ and the TF coherence function $\Gamma_{x,y}(t,f)$, shown respectively in Figs. 2(c) and (d), are again seen to be practically identical. In this example, significant coherence reduction occurs only in the TF support region of the noise; in the remainder of the TF plane there is complete coherence, thus indicating a pure linear relation between those components of $x(t)$ and $y(t)$ that are located in this "noise-free" TF region. Again, both $L_{\mathbf{\Gamma}_{x,y}^{\mathcal{G}}}(t,f)$ and $\Gamma_{x,y}(t,f)$ clearly indicate the TF dependence of coherence.

Experiment 3. We finally analyze the coherence of pressure signals $x(t)$ measured inside the cylinder of a combustion engine and vibration signals $y(t)$ measured on the engine block.[5] The goal is to see whether the pressure and vibration processes are linearly related (as assumed in [19]). Both $x(t)$ and $y(t)$ consist of several resonances with decreasing resonance frequencies. Estimates $\hat{\Gamma}_{x,y}(t,f)$ of the TF coherence function $\Gamma_{x,y}(t,f)$ are shown in Fig. 3 for two different engine speeds. (These estimates were computed using estimated Wigner-Ville spectra [20] obtained from multiple realizations.) For both engine speeds, $\big|\hat{\Gamma}_{x,y}(t,f)\big|$ is seen to be significantly larger than zero in the TF support regions of the resonances. Specifically, in the TF region of

[5]We are grateful to S. Carstens-Behrens, M. Wagner, and J. F. Böhme for providing us with the car engine data (courtesy of Aral-Forschung, Bochum).

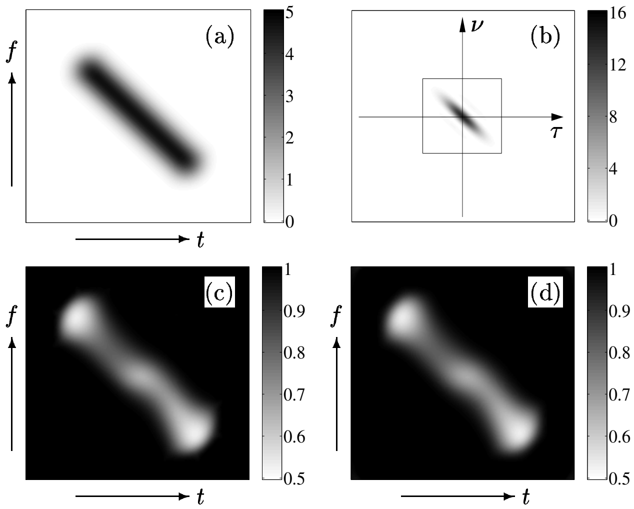

Figure 2: *Simulation results for Experiment 2: (a) WVS of $n(t)$; (b) magnitude of expected ambiguity function of $n(t)$; (c) magnitude of WS of $\mathbf{\Gamma}^{\mathcal{G}}_{x,y}$; (d) magnitude of $\Gamma_{x,y}(t,f)$. The rectangle in (b) has area 1 and allows to assess the underspread property of $n(t)$. Time duration is 256 samples; normalized frequency ranges from $-1/4$ to $1/4$.*

the first resonance the maximum of $|\hat{\Gamma}_{x,y}(t,f)|$ is about 0.9, which clearly indicates a linear relationship. For the second and third resonance, the maximum of $|\hat{\Gamma}_{x,y}(t,f)|$ is about 0.7 and 0.4, respectively. This still suggests a linear relationship, though apparently contaminated by measurement noise and extraneous interference.

7. CONCLUSIONS

We introduced and studied a coherence operator and time-frequency (TF) coherence functions for nonstationary coherence analysis. We showed that for *jointly underspread* nonstationary processes, TF coherence functions are meaningful tools for nonstationary coherence analysis. However, if the processes are not jointly underspread underspread, meaningful results can only be obtained with TF coherence functions based on *smoothed* time-varying spectra. We note that TF coherence functions can be estimated based on estimates of the time-varying spectra involved [4, 12, 13, 20]. Furthermore, many of the theorems presented can be extended to a generalized underspread concept that does not require exact compact support of spreading functions and expected ambiguity functions [5, 10, 14].

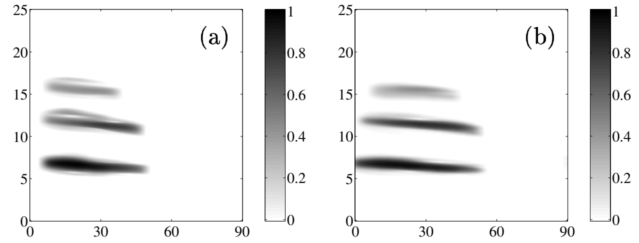

Figure 3: *Simulation results for Experiment 3: Magnitude of estimated TF coherence function $\hat{\Gamma}_{x,y}(t,f)$ at (a) 2000 rpm and (b) 3000 rpm. Horizontal axis: crank angle in degrees (proportional to time); vertical axis: frequency in kHz.*

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