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WIGNER-TYPE *a-b* AND TIME-FREQUENCY ANALYSIS BASED ON CONJUGATE OPERATORS*

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Abstract—We extend the Wigner distribution (WD) to conjugate unitary operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$. The resulting "AB-WD" is defined both as an *a-b* representation and as a time-frequency representation. Important properties and relations of the WD are generalized to the AB-WD.

1 INTRODUCTION AND OUTLINE

Recently, general frameworks for *joint a-b representations* and *time-frequency representations* based on pairs of unitary operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ have been proposed [1]-[13]. In particular, the time shift operator $\mathbf{T}_\tau$ and the frequency shift operator $\mathbf{F}_\nu$ underlying Cohen's class and the Wigner distribution (WD) [5], [14]-[17] have been generalized to the concept of *conjugate* (or *dual*) operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ [6, 8, 10, 12, 13]. Classes of *a-b* or time-frequency (TF) representations based on conjugate operators retain the structure of Cohen's class. Hence, each such class contains a central "AB-WD."

This paper introduces and discusses the AB-WD. Section 2 summarizes the theory of conjugate operators. Sections 3 and 4 introduce the AB-WD as an *a-b* and TF representation, respectively. A special case is considered in Section 5.

Cohen's class and WD. Let $x(t) \in \mathcal{L}_2(\mathbb{R})$ be a signal with Fourier transform[1] $X(f) = \int_t x(t)\, e^{-j2\pi ft}\, dt$. Cohen's class of quadratic TF representations (QTFRs) [5], [14]-[17] consists of all QTFRs $C_x(t,f)$ that are *covariant* to the time shift operator $\mathbf{T}_\tau$ and frequency shift operator $\mathbf{F}_\nu$ defined as $(\mathbf{T}_\tau\, x)(t) = x(t-\tau)$ and $(\mathbf{F}_\nu\, x)(t) = x(t)\, e^{j2\pi\nu t}$,

$$C_{\mathbf{F}_\nu \mathbf{T}_\tau x}(t,f) = C_x(t-\tau, f-\nu)\,. \tag{1}$$

Any QTFR of Cohen's class can be written as

$$C_x(t,f) = \int_{t_1}\!\int_{t_2} x(t_1)\, x^*(t_2)\, h^*(t_1-t, t_2-t)\, e^{-j2\pi f(t_1-t_2)} dt_1 dt_2 \tag{2}$$

with a 2-D kernel function $h(t_1,t_2)$. The central QTFR of Cohen's class is the WD [5], [14]-[17]

$$W_x(t,f) = \int_\tau x\Big(t+\frac{\tau}{2}\Big)\, x^*\Big(t-\frac{\tau}{2}\Big)\, e^{-j2\pi f\tau}\, d\tau \tag{3}$$

$$= \int_\nu X\Big(f+\frac{\nu}{2}\Big)\, X^*\Big(f-\frac{\nu}{2}\Big)\, e^{j2\pi t\nu}\, d\nu\,, \tag{4}$$

from which any Cohen's class QTFR can be derived as

$$C_x(t,f) = \int_{t'}\int_{f'} \psi(t-t', f-f')\, W_x(t',f')\, dt' df' \tag{5}$$

with the kernel $\psi(t,f) = \int_\tau h^*\big(-t+\frac{\tau}{2}, -t-\frac{\tau}{2}\big)\, e^{-j2\pi f\tau}\, d\tau$.

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[1] Integrals extend from $-\infty$ to ∞ unless specified otherwise.

2 CONJUGATE OPERATORS

Generalizing the shift operators $\mathbf{T}_\tau$, $\mathbf{F}_\nu$ underlying Cohen's class and the WD, we now consider two linear operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ indexed by parameters $\alpha \in \mathcal{G}$ and $\beta \in \mathcal{G}$ with $\mathcal{G} \subseteq \mathbb{R}$. These operators are assumed to be *unitary* on a linear signal space $\mathcal{X} \subseteq \mathcal{L}_2(\mathbb{R})$, and to satisfy identical *composition properties* $\mathbf{A}_{\alpha_2}\mathbf{A}_{\alpha_1} = \mathbf{A}_{\alpha_1 \bullet \alpha_2}$ and $\mathbf{B}_{\beta_2}\mathbf{B}_{\beta_1} = \mathbf{B}_{\beta_1 \bullet \beta_2}$, where $(\mathcal{G}, \bullet)$ is a commutative group [1, 6, 9, 18]. The *eigenvalues* $\lambda^A_{\alpha,b}$ and *eigenfunctions* $u^A_b(t)$ of $\mathbf{A}_\alpha$, defined by $\big(\mathbf{A}_\alpha\, u^A_b\big)(t) = \lambda^A_{\alpha,b}\, u^A_b(t)$, are indexed by a "dual" parameter b. The *A-Fourier transform* (A-FT) [1, 5, 13] is defined as

$$X_A(b) \;\triangleq\; \big\langle x, u^A_b \big\rangle = \int_t x(t)\, u^{A*}_b(t)\, dt\,. \tag{6}$$

Similar definitions apply to $\lambda^B_{\beta,a}$, $u^B_a(t)$ and the B-FT $X_B(a)$.

Two operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ as defined above are called *conjugate* [6, 10, 12] if $a \in \mathcal{G}$, $b \in \mathcal{G}$, and if application of one operator to an eigenfunction of the other operator merely produces a shift of the eigenfunction parameter, i.e., $\big(\mathbf{B}_\beta\, u^A_b\big)(t) = u^A_{b\bullet\beta}(t)$ and $\big(\mathbf{A}_\alpha\, u^B_a\big)(t) = u^B_{a\bullet\alpha}(t)$. The eigenvalues of conjugate operators can be written as [6, 10, 12]

$$\lambda^A_{\alpha,b} = e^{\pm j2\pi\,\mu(\alpha)\,\mu(b)}, \quad \lambda^B_{\beta,a} = e^{\mp j2\pi\,\mu(\beta)\,\mu(a)} = \lambda^{A*}_{\beta,a}\,, \tag{7}$$

where $\mu(g) \in \mathbb{R}$ maps $(\mathcal{G}, \bullet)$ onto $(\mathbb{R}, +)$ in the sense that $\mu(g_1 \bullet g_2) = \mu(g_1) + \mu(g_2)$, $\mu(g_0) = 0$, and $\mu(g^{-1}) = -\mu(g)$ with g_0 the identity element in $\mathcal{G}$ and g^{-1} the group-inverse of g. Due to (7), we shall write $\lambda^A_{\alpha,b} = \lambda_{\alpha,b}$ and $\lambda^B_{\beta,a} = \lambda^*_{\beta,a}$ in the following. Conjugate operators commute up to a phase factor, $\mathbf{A}_\alpha \mathbf{B}_\beta = \lambda_{\alpha,\beta}\, \mathbf{B}_\beta \mathbf{A}_\alpha$. Their eigenfunctions are related as $\big\langle u^B_a, u^A_b \big\rangle = \lambda_{a,b}$, $\int_{\mathcal{G}} u^B_a(t)\, \lambda^*_{b,a}\, d\mu(a) = u^A_b(t)$, and $\int_{\mathcal{G}} u^A_b(t)\, \lambda_{a,b}\, d\mu(b) = u^B_a(t)$, where $d\mu(g) = |\mu'(g)|\, dg$ if $\mu(g)$ is differentiable. The A-FT and B-FT are related as $X_B(a) = \int_{\mathcal{G}} X_A(b)\, \lambda^*_{a,b}\, d\mu(b)$ and $X_A(b) = \int_{\mathcal{G}} X_B(a)\, \lambda_{b,a}\, d\mu(a)$ (cf. the equivalent concept of "dual operators" in [8, 13]).

Examples. The shift operators $\mathbf{T}_\tau$, $\mathbf{F}_\nu$ underlying Cohen's class and the WD are conjugate with $(\mathcal{G}, \bullet) = (\mathbb{R}, +)$, $\mu(g) = g$, $b = f$, $a = t$, $\lambda^T_{\tau,f} = e^{-j2\pi\tau f}$, $\lambda^F_{\nu,t} = e^{j2\pi\nu t}$, $u^T_f(t) = e^{j2\pi ft}$, $u^F_t(t') = \delta(t'-t)$, $X_T(f) = X(f)$, and $X_F(t) = x(t)$. The operators $\mathbf{T}_\tau$, $\mathbf{F}_\nu$ are conjugate since $\big(\mathbf{F}_\nu\, u^T_f\big)(t) = u^T_{f+\nu}(t)$ and $\big(\mathbf{T}_\tau\, u^F_t\big)(t') = u^F_{t+\tau}(t')$. They commute up to a phase factor, $\mathbf{T}_\tau \mathbf{F}_\nu = e^{-j2\pi\tau\nu}\, \mathbf{F}_\nu \mathbf{T}_\tau$.

The operators underlying the *hyperbolic* QTFR class [19, 20] are conjugate as well, but the operators underlying the *affine class* and the *power classes* [21]-[24] are *not* conjugate.

3 AB-WD AS a-b REPRESENTATION

We now introduce[2] the *AB-WD* as an extension of the WD in (3), (4) to arbitrary *conjugate* operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$:

$$W_x^{AB}(a,b) \triangleq \int_{\mathcal{G}} X_A\left(b \bullet \beta^{1/2}\right) X_A^*\left(b \bullet \beta^{-1/2}\right) \lambda_{a,\beta}^* \, d\mu(\beta),$$

where $X_A(b)$ is defined in (6) and $\beta^{1/2}$ is defined by $\beta^{1/2} \bullet \beta^{1/2} = \beta$. The AB-WD is a function of the parameter a of $u_a^B(t)$ and the parameter b of $u_b^A(t)$ (see Section 4 for a time-frequency version of the AB-WD). It can be equivalently expressed in terms of the B-FT,

$$W_x^{AB}(a,b) = \int_{\mathcal{G}} X_B\left(a \bullet \alpha^{1/2}\right) X_B^*\left(a \bullet \alpha^{-1/2}\right) \lambda_{b,\alpha} \, d\mu(\alpha).$$

For $\mathbf{A}_\alpha = \mathbf{T}_\tau$ and $\mathbf{B}_\beta = \mathbf{F}_\nu$, these two expressions reduce to (4) and (3), respectively, so that $W_x^{TF}(a,b) = W_x(t,f)$. On the other hand, the AB-WD can be formally obtained from the WD by a unitary signal transformation and a parameter transformation $(t,f) \to (a,b)$ [10]-[12]: If $\lambda_{\alpha,b} = e^{-j2\pi\,\mu(\alpha)\,\mu(b)}$ (minus sign in the exponent), then

$$W_x^{AB}(a,b) = W_{\tilde{x}}\Big(t_r\mu(a), \frac{\mu(b)}{t_r}\Big), \quad \tilde{x}(t) = \frac{1}{\sqrt{t_r}} X_B\Big(\mu^{-1}\big(\frac{t}{t_r}\big)\Big)$$

where $t_r > 0$ is a fixed reference time and $\mu^{-1}(\cdot)$ is the function inverse to $\mu(\cdot)$. If $\lambda_{\alpha,b} = e^{j2\pi\,\mu(\alpha)\,\mu(b)}$, then

$$W_x^{AB}(a,b) = W_{\tilde{x}}\Big(t_r\mu(b), \frac{\mu(a)}{t_r}\Big), \quad \tilde{x}(t) = \frac{1}{\sqrt{t_r}} X_A\Big(\mu^{-1}\big(\frac{t}{t_r}\big)\Big).$$

Properties of the AB-WD. The properties of the AB-WD generalize the properties of the WD [5], [14]-[17]:

Real-valued: $W_x^{AB}(a,b) \in \mathbb{R}$.

Covariance property:

$$W_{\mathbf{B}_\beta \mathbf{A}_\alpha x}^{AB}(a,b) = W_x^{AB}(a \bullet \alpha^{-1}, b \bullet \beta^{-1}). \tag{8}$$

Marginal properties:

$$\int_{\mathcal{G}} W_x^{AB}(a,b)\, d\mu(b) = |X_B(a)|^2, \tag{9}$$

$$\int_{\mathcal{G}} W_x^{AB}(a,b)\, d\mu(a) = |X_A(b)|^2. \tag{10}$$

Energy distribution property:

$$\iint_{\mathcal{G}^2} W_x^{AB}(a,b)\, d\mu(a)\, d\mu(b) = \|x\|^2 = \int_t |x(t)|^2\, dt.$$

Moyal's formula/unitarity:

$$\iint_{\mathcal{G}^2} W_x^{AB}(a,b)\, W_y^{AB}(a,b)\, d\mu(a)\, d\mu(b) = \left|\langle x,y\rangle\right|^2.$$

Eigenfunction localization properties:

$$W_{u_{b_0}^A}^{AB}(a,b) = \delta\big(\mu(b \bullet b_0^{-1})\big) = \delta\big(\mu(b) - \mu(b_0)\big),$$
$$W_{u_{a_0}^B}^{AB}(a,b) = \delta\big(\mu(a \bullet a_0^{-1})\big) = \delta\big(\mu(a) - \mu(a_0)\big).$$

[2] While only the auto AB-WD will be considered for simplicity, we note that extension to the cross AB-WD is straightforward.

Interference formula:

$$\left[W_x^{AB}(a,b)\right]^2 = \iint_{\mathcal{G}^2} W_x^{AB}\left(a \bullet \alpha^{1/2}, b \bullet \beta^{1/2}\right) \cdot W_x^{AB}\left(a \bullet \alpha^{-1/2}, b \bullet \beta^{-1/2}\right) d\mu(\alpha)\, d\mu(\beta).$$

Relation with AB-AF: We next introduce the *AB-ambiguity function* (AB-AF) as

$$\begin{aligned} A_x^{AB}(\alpha,\beta) &\triangleq \left\langle \mathbf{B}_{\beta^{-1/2}}\mathbf{A}_{\alpha^{-1/2}}\, x,\, \mathbf{B}_{\beta^{1/2}}\mathbf{A}_{\alpha^{1/2}}\, x\right\rangle \\ &= \int_{\mathcal{G}} X_A\left(b \bullet \beta^{1/2}\right) X_A^*\left(b \bullet \beta^{-1/2}\right) \lambda_{\alpha,b}^*\, d\mu(b) \\ &= \int_{\mathcal{G}} X_B\left(a \bullet \alpha^{1/2}\right) X_B^*\left(a \bullet \alpha^{-1/2}\right) \lambda_{\beta,a}\, d\mu(a). \end{aligned}$$

For $\mathbf{A}_\alpha = \mathbf{T}_\tau$ and $\mathbf{B}_\beta = \mathbf{F}_\nu$, the AB-AF reduces to the conventional AF [5], [14]-[17]: $A_x^{TF}(\alpha,\beta) = A_x(\tau,\nu) = \int_t x\big(t+\frac{\tau}{2}\big)\, x^*\big(t-\frac{\tau}{2}\big)\, e^{-j2\pi\nu t}\, dt = \int_f X\big(f+\frac{\nu}{2}\big)\, X^*\big(f-\frac{\nu}{2}\big)\, e^{j2\pi\tau f}\, df$. For $\lambda_{\alpha,b} = e^{-j2\pi\,\mu(\alpha)\,\mu(b)}$ there is $A_x^{AB}(\alpha,\beta) = A_{\tilde{x}}\big(t_r\,\mu(\alpha), \mu(\beta)/t_r\big)$ with $\tilde{x}(t) = \frac{1}{\sqrt{t_r}} X_B\big(\mu^{-1}(t/t_r)\big)$, and for $\lambda_{\alpha,b} = e^{j2\pi\,\mu(\alpha)\,\mu(b)}$ there is $A_x^{AB}(\alpha,\beta) = A_{\tilde{x}}\big(t_r\,\mu(\beta), \mu(\alpha)/t_r\big)$ with $\tilde{x}(t) = \frac{1}{\sqrt{t_r}} X_A\big(\mu^{-1}(t/t_r)\big)$.

The AB-AF is related to the AB-WD as

$$A_x^{AB}(\alpha,\beta) = \iint_{\mathcal{G}^2} W_x^{AB}(a,b)\, \lambda_{\beta,a}\lambda_{\alpha,b}^*\, d\mu(a)\, d\mu(b)$$

and

$$\left|A_x^{AB}(\alpha,\beta)\right|^2 = \iint_{\mathcal{G}^2} W_x^{AB}\left(a \bullet \alpha^{1/2}, b \bullet \beta^{1/2}\right) \cdot W_x^{AB}\left(a \bullet \alpha^{-1/2}, b \bullet \beta^{-1/2}\right) d\mu(a)\, d\mu(b).$$

Uncertainty relations: We define the *A-spread* σ_x^A, *B-spread* σ_x^B, and *AB-radius* $\rho_x^{AB}(b_0)$ as $\sigma_x^{A2} \triangleq \frac{\int_{\mathcal{G}} \mu^2(b)\,|X_A(b)|^2 d\mu(b)}{\int_{\mathcal{G}} |X_A(b)|^2\, d\mu(b)}$, $\sigma_x^{B2} \triangleq \frac{\int_{\mathcal{G}} \mu^2(a)\,|X_B(a)|^2 d\mu(a)}{\int_{\mathcal{G}} |X_B(a)|^2\, d\mu(a)}$, and $\rho_x^{AB2}(b_0) \triangleq \big(\frac{\sigma_x^A}{b_0}\big)^2 + (b_0\sigma_x^B)^2$ with $b_0 \neq 0$. These quantities are related to the AB-WD as

$$\sigma_x^{A2} = \frac{\iint_{\mathcal{G}^2} \mu^2(b)\, W_x^{AB}(a,b)\, d\mu(a)d\mu(b)}{\iint_{\mathcal{G}^2} W_x^{AB}(a,b)\, d\mu(a)d\mu(b)}$$

$$\sigma_x^{B2} = \frac{\iint_{\mathcal{G}^2} \mu^2(a)\, W_x^{AB}(a,b)\, d\mu(a)d\mu(b)}{\iint_{\mathcal{G}^2} W_x^{AB}(a,b)\, d\mu(a)d\mu(b)}$$

$$\rho_x^{AB2}(b_0) = \frac{\iint_{\mathcal{G}^2} \Big[\big(\frac{\mu(b)}{b_0}\big)^2 + \big(b_0\mu(a)\big)^2\Big]\, W_x^{AB}(a,b)\, d\mu(a)d\mu(b)}{\iint_{\mathcal{G}^2} W_x^{AB}(a,b)\, d\mu(a)d\mu(b)}$$

and there exist the bounds (uncertainty relations)

$$\sigma_x^A\, \sigma_x^B \geq \frac{1}{4\pi}, \qquad \rho_x^{AB}(b_0) \geq \frac{1}{\sqrt{2\pi}} \quad \forall\, b_0 \neq 0.$$

For the next properties, we assume $\lambda_{\alpha,b} = e^{-j2\pi\,\mu(\alpha)\,\mu(b)}$. Analogous properties hold for $\lambda_{\alpha,b} = e^{j2\pi\,\mu(\alpha)\,\mu(b)}$.

Generalized chirp localization property:

$$W_x^{AB}(a,b) = \delta\big(\mu(b) - c\mu(a)\big)$$

for $X_B(a) = e^{j\pi c\mu^2(a)}$, i.e., $x(t) = \int_{\mathcal{G}} e^{j\pi c\mu^2(a)}\, u_a^B(t)\, d\mu(a)$.

Generalized instantaneous frequency property:

$$\frac{\int_{\mathcal{G}} \mu(b)\, W_x^{AB}(a,b)\, d\mu(b)}{\int_{\mathcal{G}} W_x^{AB}(a,b)\, d\mu(b)} = \frac{1}{2\pi\mu'(a)} \frac{d}{da} \arg\{X_B(a)\}.$$

Generalized group delay property:

$$\frac{\int_{\mathcal{G}} \mu(a)\, W_x^{AB}(a,b)\, d\mu(a)}{\int_{\mathcal{G}} W_x^{AB}(a,b)\, d\mu(a)} = -\frac{1}{2\pi\mu'(b)} \frac{d}{db} \arg\{X_A(b)\}.$$

Multiplication property: Let the B-FTs of $x(t)$, $y(t)$, and $z(t)$ be related as $Z_B(a) = X_B(a)\, Y_B(a)$, which can be shown to entail $Z_A(b) = \int_{\mathcal{G}} X_A(b \bullet \beta^{-1})\, Y_A(\beta)\, d\mu(\beta)$ and $z(t) = \int_{\mathcal{G}} Y_A(\beta)\, (\mathbf{B}_\beta x)(t)\, d\mu(\beta) = \int_{\mathcal{G}} X_A(\beta)\, (\mathbf{B}_\beta y)(t)\, d\mu(\beta)$. Then

$$W_z^{AB}(a,b) = \int_{\mathcal{G}} W_x^{AB}(a, b \bullet \beta^{-1})\, W_y^{AB}(a,\beta)\, d\mu(\beta).$$

Convolution property: Similarly, if $Z_B(a) = \int_{\mathcal{G}} X_B(a \bullet \alpha^{-1})$ $Y_B(\alpha)\, d\mu(\alpha)$ such that $Z_A(b) = X_A(b)\, Y_A(b)$ and $z(t) = \int_{\mathcal{G}} Y_B(\alpha)\, (\mathbf{A}_\alpha x)(t)\, d\mu(\alpha) = \int_{\mathcal{G}} X_B(\alpha)\, (\mathbf{A}_\alpha y)(t)\, d\mu(\alpha)$, then

$$W_z^{AB}(a,b) = \int_{\mathcal{G}} W_x^{AB}(a \bullet \alpha^{-1}, b)\, W_y^{AB}(\alpha, b)\, d\mu(\alpha).$$

Covariant *a*-*b* representations. The class of all quadratic *a*-*b* representations $Q_x(a,b)$ that are *covariant* to conjugate operators $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$ as (cf. (8),(1)) $Q_{\mathbf{B}_\beta \mathbf{A}_\alpha x}(a,b) = Q_x(a \bullet \alpha^{-1}, b \bullet \beta^{-1})$ can be formulated as [6, 10, 12]

$$Q_x(a,b) = \left\langle x, \mathbf{H}_{a,b}^{AB} x \right\rangle = \tag{11}$$

$$\int_{t_1}\int_{t_2} \left(\mathbf{A}_{a^{-1}}\mathbf{B}_{b^{-1}} x\right)(t_1) \left(\mathbf{A}_{a^{-1}}\mathbf{B}_{b^{-1}} x\right)^*(t_2)\, h^*(t_1,t_2)\, dt_1 dt_2$$

with $\mathbf{H}_{a,b}^{AB} = \mathbf{B}_b \mathbf{A}_a \mathbf{H}\, \mathbf{A}_{a^{-1}} \mathbf{B}_{b^{-1}}$, where $\mathbf{H}$ is an arbitrary linear operator with kernel $h(t_1,t_2)$. Equivalently,

$$Q_x(a,b) = \iint_{\mathcal{G}^2} \Psi(\alpha,\beta)\, A_x^{AB}(\alpha,\beta)\, \lambda_{\beta,a}^* \lambda_{\alpha,b}\, d\mu(\alpha)\, d\mu(\beta) \tag{12}$$

where the kernel $\Psi(\alpha,\beta)$ is related to $h(t_1,t_2)$ [6, 10, 12]. The covariant class $\{Q_x(a,b)\}$ is the extension of Cohen's class in (2) to arbitrary conjugate operators $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$. In particular, the AB-WD is obtained with $\Psi(\alpha,\beta) \equiv 1$.

It can be shown that the covariant class $\{Q_x(a,b)\}$ can be derived from the AB-WD $W_x^{AB}(a,b)$ as (cf. (5))

$$Q_x(a,b) = \iint_{\mathcal{G}^2} \psi\left(a \bullet \alpha^{-1}, b \bullet \beta^{-1}\right) W_x^{AB}(\alpha,\beta)\, d\mu(\alpha)\, d\mu(\beta), \tag{13}$$

where the kernel $\psi(a,b)$ is related to the kernel $\Psi(\alpha,\beta)$ in (12) as $\psi(a,b) = \iint_{\mathcal{G}^2} \Psi(\alpha,\beta)\, \lambda_{\beta,a}^* \lambda_{\alpha,b}\, d\mu(\alpha)\, d\mu(\beta)$.

AB-spectrogram. Setting $h(t_1,t_2) = g(t_1)\, g^*(t_2)$ in (11) yields the *AB-spectrogram*

$$S_x^{AB}(a,b) = \left|L_x^{AB}(a,b)\right|^2 \quad \text{with} \quad L_x^{AB}(a,b) = \left\langle x, \mathbf{B}_b \mathbf{A}_a g\right\rangle.$$

Here, $\psi(a,b) = W_g^{AB}(a^{-1}, b^{-1})$ so that $S_x^{AB}(a,b) = \iint_{\mathcal{G}^2} W_g^{AB}\left(\alpha \bullet a^{-1}, \beta \bullet b^{-1}\right) W_x^{AB}(\alpha,\beta)\, d\mu(\alpha)\, d\mu(\beta)$.

4 AB-WD AS TF REPRESENTATION

The AB-WD can be re-formulated as a quadratic time-frequency (TF) representation. Let $\nu_b^A(t)$ denote the instantaneous frequency of the eigenfunction $u_b^A(t)$, and let $\tau_a^B(f)$ denote the group delay of the eigenfunction $u_a^B(t)$. For any $(a,b) \in \mathcal{G}^2$, the corresponding functions $\nu_b^A(t)$ and $\tau_a^B(f)$ are assumed[3] to intersect in a unique TF point (t,f). Hence, there is a one-to-one correspondence $(t,f) = l(a,b)$, $(a,b) = l^{-1}(t,f)$ where $l(\cdot,\cdot)$ will be called the *localization function* of the operator pair $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$. The localization function is constructed by solving the system of equations $\nu_b^A(t) = f$, $\tau_a^B(f) = t$ for (t,f) given (a,b) [11, 3, 6].

The TF version of the AB-WD is now defined as

$$\widetilde{W}_x^{AB}(t,f) \triangleq W_x^{AB}(a,b)\Big|_{(a,b)=l^{-1}(t,f)}.$$

All properties of the *a*-*b* version of the AB-WD (see Section 3) can be re-formulated for the TF version of the AB-WD. For example, the TF version of the covariance (8) reads

$$\widetilde{W}_{\mathbf{B}_\beta \mathbf{A}_\alpha x}^{AB}(t,f) = \widetilde{W}_x^{AB}\Big(l\left(l^{-1}(t,f) \circ (\alpha^{-1},\beta^{-1})\right)\Big), \tag{14}$$

where $(a,b) \circ (\alpha,\beta) \triangleq (a \bullet \alpha, b \bullet \beta)$. The marginal properties (9), (10) become

$$\int_f \widetilde{W}_x^{AB}\left(\tau_a^B(f), f\right) d\tilde{\mu}_1(f;a) = |X_B(a)|^2, \tag{15}$$

$$\int_t \widetilde{W}_x^{AB}\left(t, \nu_b^A(t)\right) d\tilde{\mu}_2(t;b) = |X_A(b)|^2, \tag{16}$$

where $d\tilde{\mu}_1(f;a)$ and $d\tilde{\mu}_2(t;b)$ follow from $\tau_a^B(\cdot)$, $\nu_b^A(\cdot)$, and $d\mu(\cdot)$. All other properties and relations listed in Section 3 can be transferred to the TF domain as well.

5 AN EXAMPLE

We shall finally consider an example. Let the operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ be defined on the space $\mathcal{X} = \mathcal{L}_2(\mathbb{R}_+)$ as

$$(\mathbf{A}_\alpha x)(t) = \frac{1}{\sqrt{\alpha}}\, x\left(\frac{t}{\alpha}\right), \qquad (\mathbf{B}_\beta\, x)(t) = e^{j2\pi \ln\beta\, \ln(t/t_r)}\, x(t)$$

where $t, \alpha, \beta > 0$ with $t_r > 0$ fixed. Since $\mathbf{A}_{\alpha_2}\mathbf{A}_{\alpha_1} = \mathbf{A}_{\alpha_1\alpha_2}$ and $\mathbf{B}_{\beta_2}\mathbf{B}_{\beta_1} = \mathbf{B}_{\beta_1\beta_2}$, the underlying group is the multiplicative group, $(\mathcal{G}, \bullet) = (\mathbb{R}_+, \cdot)$, with identity element $g_0 = 1$ and inverse elements $g^{-1} = 1/g$. The eigenvalues/functions of $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are $\lambda_{\alpha,b}^A = e^{-j2\pi \ln\alpha \ln b}$, $u_b^A(t) = \frac{1}{\sqrt{t}}\, e^{j2\pi \ln b\, \ln(t/t_r)}$ and $\lambda_{\beta,a}^B = e^{j2\pi \ln\beta \ln a}$, $u_a^B(t) = \frac{1}{\sqrt{t_r a}}\, \delta(\ln\frac{t}{t_r} - \ln a)$. Note that $\mu(g) = \ln g$ and $d\mu(g) = \frac{dg}{g}$. The A-FT is the Mellin-type transform $X_A(b) = \int_0^\infty x(t)\, e^{-j2\pi \ln b\, \ln(t/t_r)}\, \frac{dt}{\sqrt{t}}$ and the B-FT is $X_B(a) = \sqrt{t_r a}\;\, x(t_r a)$. The operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are *conjugate*, $\left(\mathbf{B}_\beta\, u_b^A\right)(t) = u_{b\beta}^A(t)$ and $\left(\mathbf{A}_\alpha\, u_a^B\right)(t) = u_{a\alpha}^B(t)$. They commute up to a phase factor, $\mathbf{A}_\alpha \mathbf{B}_\beta = e^{-j2\pi \ln\alpha \ln\beta}\, \mathbf{B}_\beta \mathbf{A}_\alpha$.

The *a*-*b* version of the AB-WD is the *a*-*b*, time-domain version of the *Q-distribution* [19, 25, 26]

$$W_x^{AB}(a,b) = t_r a \int_{-\infty}^{\infty} x\left(t_r a\, e^{u/2}\right) x^*\left(t_r a\, e^{-u/2}\right) e^{-j2\pi(\ln b)u} du$$

[3] In certain cases, this assumption holds if one uses the group delay of $u_b^A(t)$ and the instantaneous frequency of $u_a^B(t)$; here, an analogous theory can be formulated.

with $a,b>0$. It satisfies the covariance property

$$W^{AB}_{\mathbf{B}_\beta \mathbf{A}_\alpha x}(a,b) \;=\; W^{AB}_x\!\left(\frac{a}{\alpha},\frac{b}{\beta}\right),$$

the marginal properties

$$\int_0^\infty W^{AB}_x(a,b)\,\frac{db}{b} \;=\; t_r a\,|x(t_r a)|^2\,,$$

$$\int_0^\infty W^{AB}_x(a,b)\,\frac{da}{a} \;=\; \left|\int_0^\infty x(t)\,e^{-j2\pi \ln b\,\ln(t/t_r)}\,\frac{dt}{\sqrt{t}}\right|^2,$$

and other properties (cf. Section 3). The covariant class (11) is the a-b, time-domain version of the *hyperbolic class* [19, 20]

$$Q_x(a,b) \;=$$
$$\frac{1}{a}\int_0^\infty\!\!\int_0^\infty x(t_1)\,x^*(t_2)\,h^*\!\left(\frac{t_1}{a},\frac{t_2}{a}\right)e^{-j2\pi(\ln b)\,\ln(t_1/t_2)}\,dt_1dt_2\,;$$

it can be derived from $W^{AB}_x(a,b)$ as (see (13))

$$Q_x(a,b) = \int_0^\infty\!\!\int_0^\infty \psi\!\left(\frac{a}{\alpha},\frac{b}{\beta}\right)W^{AB}_x(\alpha,\beta)\,\frac{d\alpha}{\alpha}\frac{d\beta}{\beta}\,.$$

With $\nu^A_b(t) = (\ln b)/t$ and $\tau^B_a(f) \equiv t_r a$, the localization function is obtained as $(t,f) = l(a,b) = \left(t_r a, \frac{\ln b}{t_r a}\right)$ with inverse $(a,b) = l^{-1}(t,f) = \left(\frac{t}{t_r}, e^{tf}\right)$. The TF version of the AB-WD is then [19, 25, 26]

$$\widetilde{W}^{AB}_x(t,f) \;=\; W^{AB}_x\!\left(\frac{t}{t_r}, e^{tf}\right)$$
$$\;=\; t\int_{-\infty}^{\infty} x(t\,e^{u/2})\,x^*(t\,e^{-u/2})\,e^{-j2\pi t f u}\,du$$

for $t>0$; it satisfies the covariance property (cf. (14))

$$\widetilde{W}^{AB}_{\mathbf{B}_\beta \mathbf{A}_\alpha x}(t,f) \;=\; \widetilde{W}^{AB}_x\!\left(\frac{t}{\alpha},\,\alpha\!\left(f-\frac{\ln\beta}{t}\right)\right) \qquad (17)$$

and the marginal properties (cf. (15), (16))

$$\int_{-\infty}^{\infty} \widetilde{W}^{AB}_x(t,f)\,df \;=\; |x(t)|^2\,,$$

$$\int_0^\infty \widetilde{W}^{AB}_x\!\left(t,\frac{\ln b}{t}\right)\frac{dt}{t} \;=\; \left|\int_0^\infty x(t)\,e^{-j2\pi \ln b\,\ln(t/t_r)}\,\frac{dt}{\sqrt{t}}\right|^2.$$

The class of QTFRs satisfying the covariance (17) is [19, 20]

$$\tilde{Q}_x(t,f) \;=\; Q_x\!\left(\frac{t}{t_r}, e^{tf}\right) \;=$$
$$\frac{t_r}{t}\int_0^\infty\!\!\int_0^\infty x(t_1)\,x^*(t_2)\,h^*\!\left(t_r\frac{t_1}{t}, t_r\frac{t_2}{t}\right)e^{-j2\pi t f\,\ln(t_1/t_2)}\,dt_1dt_2\,;$$

it can be derived from $\widetilde{W}^{AB}_x(t,f)$ as

$$\tilde{Q}_x(t,f) = \int_{t'=0}^{\infty}\int_{f'=-\infty}^{\infty} \psi\!\left(\frac{t}{t'}, e^{tf-t'f'}\right)\widetilde{W}^{AB}_x(t',f')\,dt'df'.$$

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