# A Formulary

Contents

## A.1 Continuous-time Fourier transform

### Relations

| | | |
|---|---|---|
| $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega$ | $\Longleftrightarrow$ | $X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$ |
| $x(t-T_0)$ | $\Longleftrightarrow$ | $e^{-j\omega T_0} X(j\omega)$ |
| $e^{j\omega_0 t} x(t)$ | $\Longleftrightarrow$ | $X(j(\omega-\omega_0))$ |
| $x^*(t)$ | $\Longleftrightarrow$ | $X^*(-j\omega)$ |
| $x(-t)$ | $\Longleftrightarrow$ | $X(-j\omega)$ |
| $x(at)$ | $\Longleftrightarrow$ | $\frac{1}{\lvert a\rvert} X\left(\frac{j\omega}{a}\right)$ |
| $(x*y)(t)$ | $\Longleftrightarrow$ | $X(j\omega)Y(j\omega)$ |
| $x(t)y(t)$ | $\Longleftrightarrow$ | $\frac{1}{2\pi}(X*Y)(j\omega)$ |
| $x_e(t) = \frac{1}{2}(x(t)+x^*(-t))$ | $\Longleftrightarrow$ | $\Re e\{X(j\omega)\}$ |
| $x_o(t) = \frac{1}{2}(x(t)-x^*(-t))$ | $\Longleftrightarrow$ | $j\Im m\{X(j\omega)\}$ |
| $\Re e\{x(t)\}$ | $\Longleftrightarrow$ | $X_e(j\omega) = \frac{1}{2}(X(j\omega)+X^*(-j\omega))$ |
| $j\Im m\{x(t)\}$ | $\Longleftrightarrow$ | $X_o(j\omega) = \frac{1}{2}(X(j\omega)-X^*(-j\omega))$ |
| $tx(t)$ | $\Longleftrightarrow$ | $j\frac{dX(j\omega)}{d\omega}$ |
| $\frac{d^n x(t)}{dt^n}$ | $\Longleftrightarrow$ | $(j\omega)^n X(j\omega)$ |
| $\int_{-\infty}^{t} x(\tau)\, d\tau$ | $\Longleftrightarrow$ | $\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$ |

### Transform pairs

| | | |
|---|---|---|
| $\delta(t-T_0)$ | $\Longleftrightarrow$ | $e^{-j\omega T_0}$ |
| $e^{j\omega_0 t}$ | $\Longleftrightarrow$ | $2\pi\delta(\omega-\omega_0)$ |
| $\cos\omega_0 t$ | $\Longleftrightarrow$ | $\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$ |
| $\sin\omega_0 t$ | $\Longleftrightarrow$ | $\frac{\pi}{j}\delta(\omega-\omega_0)-\frac{\pi}{j}\delta(\omega+\omega_0)$ |
| $\sum_{k=-\infty}^{\infty} \delta(t-kT)$ | $\Longleftrightarrow$ | $\frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$ |
| $\sum_{k=-\infty}^{\infty} c_k\, e^{j\frac{2\pi k}{T}t}$ | $\Longleftrightarrow$ | $2\pi\sum_{k=-\infty}^{\infty} c_k\,\delta\left(\omega - \frac{2\pi k}{T}\right)$ |
| $\sigma(t)$ | $\Longleftrightarrow$ | $\frac{1}{j\omega}+\pi\delta(\omega)$ |

$$\text{sign}(t) \iff \frac{2}{j\omega}$$

$$e^{-at}\sigma(t), \quad \Re e\{a\} > 0 \iff \frac{1}{a + j\omega}$$

$$\frac{t^{n-1}}{(n-1)!}\, e^{-at}\sigma(t), \quad \Re e\{a\} > 0 \iff \frac{1}{(a + j\omega)^n}$$

$$e^{-a|t|}, \quad \Re e\{a\} > 0 \iff \frac{2a}{a^2 + \omega^2}$$

$$\frac{\sin \omega_c t}{\pi t} \iff X(j\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

$$x(t) = \begin{cases} 1, & |t| \le T_1 \\ 0, & |t| > T_1 \end{cases} \iff 2\,\frac{\sin \omega T_1}{\omega}$$

$$x(t) = \begin{cases} 1 - \frac{|t|}{T_1}, & |t| \le T_1 \\ 0, & |t| > T_1 \end{cases} \iff 4\,\frac{\sin^2 \frac{\omega T_1}{2}}{T_1 \omega^2}$$

$$e^{-at^2}, \quad a > 0 \iff \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}$$

$$\frac{d^n \delta(t)}{dt^n} \iff (j\omega)^n$$

$$t^n \iff 2\pi j^n \frac{d^n \delta(\omega)}{d\omega^n}$$

$$|t| \iff -\frac{2}{\omega^2}$$

## A.2 Discrete-time Fourier transform

### Relations

$$x[n] = \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\theta})\, e^{j\theta n} d\theta \iff X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\theta n}$$

$$x[n - N_0] \iff e^{-j\theta N_0} X(e^{j\theta})$$

$$e^{j\theta_0 n} x[n] \iff X(e^{j(\theta - \theta_0)})$$

$$x^*[n] \iff X^*(e^{-j\theta})$$

$$x[-n] \iff X(e^{-j\theta})$$

$$(x * y)[n] \iff X(e^{j\theta}) Y(e^{j\theta})$$

$$x[n]y[n] \iff \tfrac{1}{2\pi}(X * Y)(e^{j\theta})$$

$$x_e[n] = \tfrac{1}{2}(x[n] + x^*[-n]) \iff \Re e\{X(e^{j\theta})\}$$

$$x_o[n] = \tfrac{1}{2}(x[n] - x^*[-n]) \iff j\Im m\{X(e^{j\theta})\}$$

$$\Re e\{x[n]\} \iff X_e(e^{j\theta}) = \tfrac{1}{2}(X(e^{j\theta}) + X^*(e^{-j\theta}))$$

$$j\Im m\{x[n]\} \iff X_o(e^{j\theta}) = \tfrac{1}{2}(X(e^{j\theta}) - X^*(e^{-j\theta}))$$

$$nx[n] \iff j\frac{dX(e^{j\theta})}{d\theta}$$

$$\sum_{k=-\infty}^{n} x[k] \iff \frac{1}{1-e^{-j\theta}}X(e^{j\theta}) + \pi X(e^{j0})\delta_{2\pi}(\theta)$$

**Transform pairs**

Recall that the discrete-time Fourier transform is $2\pi$ periodic in the frequency domain; in particular, $\delta_{2\pi}(\theta) = \sum_{k=-\infty}^{\infty} \delta(\theta - 2\pi k)$.

$$\delta[n - N_0] \iff e^{-j\theta N_0}$$

$$e^{j\theta_0 n} \iff 2\pi\delta_{2\pi}(\theta - \theta_0)$$

$$\cos\theta_0 n \iff \pi\delta_{2\pi}(\theta - \theta_0) + \pi\delta_{2\pi}(\theta + \theta_0)$$

$$\sin\theta_0 n \iff \tfrac{\pi}{j}\delta_{2\pi}(\theta - \theta_0) - \tfrac{\pi}{j}\delta_{2\pi}(\theta + \theta_0)$$

$$\sum_{k=-\infty}^{\infty} \delta[n - kN] \iff \frac{2\pi}{N}\sum_{k=-\infty}^{\infty} \delta\left(\theta - \frac{2\pi k}{N}\right)$$

$$\sigma[n] \iff \frac{1}{1-e^{-j\theta}} + \pi\delta_{2\pi}(\theta)$$

$$a^n\sigma[n], \quad |a| < 1 \iff \frac{1}{1-ae^{-j\theta}}$$

$$\frac{\sin\alpha n}{\pi n}, \quad 0 < \alpha < \pi \iff X(e^{j\theta}) = \begin{cases} 1 & 0 \le |\theta| \le \alpha \\ 0 & \alpha < |\theta| < \pi \end{cases}$$

$$x[n] = \begin{cases} 1 & 0 \le |n| \le N_1 \\ 0 & 0 \le |n| > N_1 \end{cases} \iff \frac{\sin\left((2N_1+1)\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}}$$

## A.3 $z$ transform

Here, the third column states the region of convergence (ROC).

**Relations**

$$x[n] = \frac{1}{2\pi j}\oint_{\mathcal{C}} X(z)z^{n-1}dz \iff X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \qquad R_{x_-} < |z| < R_{x_+}$$

| | | | |
|---|---|---|---|
| $x[n]$ | $\Longleftrightarrow$ | $X(z)$ | $R_{x_-} < \|z\| < R_{x_+}$ |
| $ax[n] + by[n]$ | $\Longleftrightarrow$ | $aX(z) + bY(z)$ | $\max(R_{x_-}, R_{y_-}) < \|z\| < \min(R_{x_+}, R_{y_+})$ |
| $x[n + n_0]$ | $\Longleftrightarrow$ | $z^{n_0} X(z)$ | $R_{x_-} < \|z\| < R_{x_+}$ |
| $z_0^n x[n]$ | $\Longleftrightarrow$ | $X\left(\frac{z}{z_0}\right)$ | $\|z_0\| R_{x_-} < \|z\| < \|z_0\| R_{x_+}$ |
| $nx[n]$ | $\Longleftrightarrow$ | $-z\,\frac{dX(z)}{dz}$ | $R_{x_-} < \|z\| < R_{x_+}$ |
| $x^*[n]$ | $\Longleftrightarrow$ | $X^*(z^*)$ | $R_{x_-} < \|z\| < R_{x_+}$ |
| $x[-n]$ | $\Longleftrightarrow$ | $X(z^{-1})$ | $\frac{1}{R_{x_+}} < \|z\| < \frac{1}{R_{x_-}}$ |
| $\Re e\{x[n]\}$ | $\Longleftrightarrow$ | $\frac{1}{2}(X(z) + X^*(z^*))$ | $R_{x_-} < \|z\| < R_{x_+}$ |
| $j\Im m\{x[n]\}$ | $\Longleftrightarrow$ | $\frac{1}{2}(X(z) - X^*(z^*))$ | $R_{x_-} < \|z\| < R_{x_+}$ |
| $(x * y)[n]$ | $\Longleftrightarrow$ | $X(z)Y(z)$ | $\max(R_{x_-}, R_{y_-}) < \|z\| < \min(R_{x_+}, R_{y_+})$ |
| $x[n]\,y[n]$ | $\Longleftrightarrow$ | $\frac{1}{2\pi j}\oint_{\mathcal{C}} \frac{X(v)}{v} Y\left(\frac{z}{v}\right)\,dv$ | $R_{x_-}R_{y_-} < \|z\| < R_{x_+}R_{y_+}$ |
| $\sum_{k=-\infty}^{n} x[k]$ | $\Longleftrightarrow$ | $\frac{1}{1 - z^{-1}}\,X(z)$ | $\max(R_{x_-}, 1) < \|z\| < R_{x_+}$ |

**Transform pairs**

| | | | |
|---|---|---|---|
| $\delta[n]$ | $\Longleftrightarrow$ | $1$ | $\forall z$ |
| $\sigma[n]$ | $\Longleftrightarrow$ | $\frac{z}{z-1}$ | $\|z\| > 1$ |
| $-\sigma[-n-1]$ | $\Longleftrightarrow$ | $\frac{z}{z-1}$ | $\|z\| < 1$ |
| $\alpha^n \sigma[n]$ | $\Longleftrightarrow$ | $\frac{z}{z-\alpha}$ | $\|z\| > \|\alpha\|$ |
| $-\alpha^n \sigma[-n-1]$ | $\Longleftrightarrow$ | $\frac{z}{z-\alpha}$ | $\|z\| < \|\alpha\|$ |
| $n\sigma[n]$ | $\Longleftrightarrow$ | $\frac{z}{(z-1)^2}$ | $\|z\| > 1$ |
| $-n\sigma[-n-1]$ | $\Longleftrightarrow$ | $\frac{z}{(z-1)^2}$ | $\|z\| < 1$ |
| $\sin\alpha n\ \sigma[n]$ | $\Longleftrightarrow$ | $\frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}$ | $\|z\| > 1$ |
| $\cos\alpha n\ \sigma[n]$ | $\Longleftrightarrow$ | $\frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$ | $\|z\| > 1$ |
| $\rho^n \sin\alpha n\ \sigma[n]$ | $\Longleftrightarrow$ | $\frac{\rho z\sin\alpha}{z^2 - 2\rho z\cos\alpha + \rho^2}$ | $\|z\| > \rho$ |

| | | | |
|---|---|---|---|
| $\rho^n \cos \alpha n \; \sigma[n]$ | $\Longleftrightarrow$ | $\dfrac{z(z - \rho \cos \alpha)}{z^2 - 2\rho z \cos \alpha + \rho^2}$ | $\lvert z\rvert > \rho$ |
| $\sin(\alpha n + \varphi) \; \sigma[n]$ | $\Longleftrightarrow$ | $\dfrac{z^2 \sin \varphi + z \sin(\alpha - \varphi)}{z^2 - 2z \cos \alpha + 1}$ | $\lvert z\rvert > 1$ |
| $\dfrac{1}{n}, \quad n > 0$ | $\Longleftrightarrow$ | $\log_e \dfrac{z}{z-1}$ | $\lvert z\rvert > 1$ |
| $\dfrac{1 - e^{-\alpha n}}{n} \; \sigma[n]$ | $\Longleftrightarrow$ | $\alpha + \log_e \dfrac{z - e^{-\alpha}}{z - 1}$ | $\lvert z\rvert > 1, \; \alpha > 0$ |
| $\dfrac{\sin \alpha n}{n} \; \sigma[n]$ | $\Longleftrightarrow$ | $\alpha + \arctan \left( \dfrac{\sin \alpha}{z - \cos \alpha} \right)$ | $\lvert z\rvert > \cos \alpha, \; \alpha > 0$ |