Name:

You are allowed to use your own copy of the lecture notes, a formulary and a calculator. Any other documents, especially pre-calculated examples, are forbidden.

Problem 1 (15 points) Consider a random variable X with "triangular" probability density function (pdf)

$$f_X(x) = \begin{cases} 2 \cdot (1-x), & \text{if } 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

a) Sketch the pdf of X.

b) Calculate the mean and variance of X.

c) Suppose X_1 and X_2 are statistically independent random variables and each has the same pdf $f_{X_1}(x) = f_{X_2}(x) = f_X(x)$. Find the pdf $f_Y(y)$ of $Y = X_1 + X_2$.

d) Calculate the mean and variance of Y without using its pdf $f_Y(y)$, calculated in c).

e) Find the probability $\mathrm{P}\{Y > 1\}$.

Problem 2 (20 points) Consider a Gaussian distributed random variable $A \sim \mathcal{N}(0, \sigma^2)$ and a discrete random variable $B \in \{-1, 1\}$ with $\mathrm{P}\{B = 1\} = \mathrm{P}\{B = -1\} = 1/2$, i.e., $p_B(b) = 1/2[\delta(b-1) + \delta(b+1)]$. The random variables A and B are statistically independent.

a) Calculate the joint probability density function (pdf) $f_{A,W}(a, w)$ for $W = AB$.

b) Calculate the marginal pdf $f_W(w)$.

c) Find out whether W and A are statistically independent and/or uncorrelated and/or orthogonal. Justify your answer.

d) Repeat subtasks a) – c) for the case:
A is a discrete random variable with probability $\mathrm{P}\{A = 1\} = \mathrm{P}\{A = -1\} = 1/2$ and B is a Gaussian distributed random variable with $\mathcal{N}(0, \sigma^2)$.

Problem 3 (15 points) Consider the random processes

$$\begin{aligned} Y_1[n] &= A[n]\cos(\theta_0 n) \\ Y_2[n] &= A[n]\cos(\theta_0 n + \Phi) \\ Y_3[n] &= A[n]\cos(\theta_0 n) + B[n]\sin(\theta_0 n), \end{aligned}$$

where θ_0 is a constant frequency. The random processes $A[n]$ and $B[n]$ are jointly WSS and the random variable Φ is uniformly distributed in the interval $[-\pi, \pi[$ and statistically independent of $A[n]$.

a) Calculate the means of $Y_1[n], Y_2[n]$, and $Y_3[n]$.

b) Calculate the autocorrelation functions of $Y_1[n], Y_2[n]$, and $Y_3[n]$.

c) Are $Y_1[n]$ and $Y_2[n]$ WSS or wide-sense cyclostationary? Justify your answer. Calculate the Period N_0 if the random processes are wide-sense cyclostationary.

d) Under which conditions for $A[n]$ and $B[n]$ is $Y_3[n]$ WSS?

Problem 4 (20 points) Consider a zero-mean random vector $\underline{X} = \begin{pmatrix} X_1 & X_2 \end{pmatrix}^T$ corrupted by a zero-mean noise vector $\underline{Y} = \begin{pmatrix} Y_1 & Y_2 \end{pmatrix}^T$ that is statistically independent of $\underline{X}$. At the receiver, the vector $\underline{Z} = \underline{X} + \underline{Y}$ is observed. Let

$$\underline{\underline{R}}_{\underline{X}} = \begin{pmatrix} 2.5 & \alpha \\ \beta & 2.5 \end{pmatrix}, \qquad \underline{\underline{R}}_{\underline{Y}} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

a) Specify β in terms of α and find the possible value ranges for α and β.

b) For $\alpha = 1$, find the homogeneous LMMSE estimator of $\underline{X}$.

c) Find the corresponding minimum mean square error $\varepsilon_{\min}$.

d) For arbitrary α, find the eigenvalues λ_k and orthonormal eigenvectors $\underline{u}_k$ of $\underline{\underline{R}}_{\underline{X}}$. Show that for $\underline{\underline{U}} = \begin{pmatrix} \underline{u}_1 & \underline{u}_2 \end{pmatrix}$, the random vector $\underline{W} = \underline{\underline{U}}^T \underline{X}$ consists of orthogonal elements.

e) Calculate the correlation matrix of the random vector $\underline{V} = \underline{\underline{U}}\,\underline{Z}$. Are the elements of $\underline{V}$ orthogonal?