Name:

> You are allowed to use your own copy of the lecture notes, a formulary and a calculator. Any other documents, especially pre-calculated examples, are forbidden.

Problem 1 (15.5) Let $\mathsf{y}[n]$ be a discrete random process defined as $\mathsf{y}[2n] = \mathsf{y}_1[n]$, $\mathsf{y}[2n+1] = \mathsf{y}_2[n]$. Here,

$$\mathsf{y}_1[n] = 1 + \mathsf{v}_1[n], \quad \mathsf{y}_2[n] = \mathsf{a}[n] + \mathsf{v}_2[n],$$

where $\mathsf{v}_1[n], \mathsf{v}_2[n] \sim \mathcal{N}(0, \sigma^2)$ are identically distributed and statistically independent. $\mathsf{a}[n]$ is statistically independent from $\mathsf{v}_1[n]$ and $\mathsf{v}_2[n]$ and takes the values 1 and -1 with equal probability for any n. The random processes $\mathsf{v}_1[n], \mathsf{v}_2[n]$, and $\mathsf{a}[n]$ are jointly strict-sense stationary (jointly SSS).

a) Find the joint conditional pdf $f_{\mathsf{y}_1[n],\mathsf{y}_2[n]|\mathsf{a}[n]}(y_1, y_2|1)$ of the random variables $\mathsf{y}_1[n], \mathsf{y}_2[n]$.

b) Find the joint pdf $f_{\mathsf{y}_1[n],\mathsf{y}_2[n]}(y_1, y_2)$ of the random variables $\mathsf{y}_1[n], \mathsf{y}_2[n]$.

c) Are the two random variables $\mathsf{y}_1[n], \mathsf{y}_2[n]$ statistically independent? Justify your answer.

d) Find the means $\mu_{\mathsf{y}_1}[n]$ and $\mu_{\mathsf{y}_2}[n]$ and express the autocorrelation functions $R_{\mathsf{y}_1}[n_1, n_2]$ and $R_{\mathsf{y}_2}[n_1, n_2]$ in terms of the autocorrelation functions of the strict-sense stationary processes $r_{\mathsf{v}_1}[m], r_{\mathsf{v}_2}[m]$, and $r_{\mathsf{a}}[m]$, where $m = n_1 - n_2$.

e) Is $\mathsf{y}[n]$ wide-sense stationary? Justify your answer.

Problem 2 (20.5 points) Consider the joint probability density function (pdf)

$$f_{\mathsf{x},\mathsf{y}}(x,y) = \begin{cases} K, & x^2+y^2 \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

The random variables x and y are mapped onto two new random variables $\mathsf{z} = \mathsf{x}^2$ and $\mathsf{r} = \sqrt{\mathsf{x}^2+\mathsf{y}^2}$.

a) Sketch the joint pdf $f_{\mathsf{x},\mathsf{y}}(x,y)$ and calculate the constant K.

b) Calculate the expectations $\mathrm{E}\{\mathsf{r}\}$ and $\mathrm{E}\{\mathsf{r}^2\}$.

c) Calculate the probability $\mathrm{P}\{\mathsf{x}^2+\mathsf{y}^2 < 1/2\}$.

d) Calculate the marginal pdfs $f_{\mathsf{x}}(x)$ and $f_{\mathsf{z}}(z)$.

e) Calculate the mixed moment $m_{\mathsf{xy}}^{(1,1)}$.

f) Are x and y uncorrelated and/or statistically independent? Justify your answer.

Hints:

- $\int_0^1 \sqrt{1-u^2}du = \frac{\pi}{4}$
- Conversion between cartesian coordinates and polar coordinates: $dxdy = rdrd\phi$

Problem 3 (15 points) Consider a random variable x with "triangular" probability density function (pdf)

$$f_{\mathsf{x}}(x) = \begin{cases} 2\cdot(1-x), & \text{if } 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

a) Sketch the pdf of x.

b) Calculate the mean and variance of x.

c) Suppose x_1 and x_2 are statistically independent random variables and each has the same pdf $f_{\mathsf{x}_1}(x) = f_{\mathsf{x}_2}(x) = f_{\mathsf{x}}(x)$. Find the pdf $f_{\mathsf{y}}(y)$ of $\mathsf{y} = \mathsf{x}_1 + \mathsf{x}_2$.

d) Calculate the mean and variance of y without using its pdf $f_{\mathsf{y}}(y)$, calculated in c).

e) Find the probability $\mathrm{P}\{Y > 1\}$.

Problem 4 (19 points) Consider a random vector $\mathsf{v} = (\mathsf{v}_1 \;\; \mathsf{v}_2)^T$ with $\sigma^2_{\mathsf{v}_1} = \sigma^2_{\mathsf{v}_2} = 2$, $\mu_{\mathsf{v}_1} = \mu_{\mathsf{v}_2} = 3$, $R_{\mathsf{v}_1,\mathsf{v}_2} = \alpha$ and $R_{\mathsf{v}_2,\mathsf{v}_1} = \beta$.

a) Specify β in terms of α and find the possible value ranges for α and β.

b) For $\alpha = 8$, specify the correlation matrix $\boldsymbol{R}_{\mathsf{v}}$ and the covariance matrix $\boldsymbol{C}_{\mathsf{v}}$.

c) Find the eigenvalues λ_1 and λ_2 of $\boldsymbol{C}_{\mathsf{v}}$ for $\alpha = 8$.

d) Find the corresponding orthonormal eigenvectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

e) Find the whitening (decorrelation) transformation matrix $\boldsymbol{A}$.

f) Show that the resulting elements of the random vector $\mathsf{w} = \boldsymbol{A}\mathsf{v}$ are uncorrelated.