Name: ______________________________

Matriculation number: ______________________

Study code: ◯ 066 437 (master TC), ◯ 033 235 (bachelor EE), ◯ other: ________

Instructions (READ CAREFULLY)

- A copy of the lecture notes is provided. You may use these lecture notes and a simple calculator during the exam. The use of any other documents or electronic devices is not permitted.
- Begin your solution to each problem on a new sheet of paper. If you do not attempt a problem, turn in a blank page on which you have written the missing problem number.
- Be sure to hand in your solutions by arranging them in such a way that the problems 1-4 appear in order.
- If you arrive at a result which is obviously incorrect, indicate that you are aware that the result is incorrect and elaborate, if possible.
- If the answer to one part depends upon the results of earlier parts that you were not able to answer, demonstrate your competence on the remaining parts by making reasonable assumptions about answers to the missing parts.
- Note that the problems are weighted differently. Start with the problem that you can solve quickest.
- Legible writing and a clear layout of your derivations and solutions are absolutely necessary.
- The result of this exam will be sent to your generic student email address (e<matriculation_number>@student.tuwien.ac.at) within two weeks.

Problem	1	2	3	4	Sum
Max. credits	18	18	19	15	70
Credits					

Exercise credits: **Total credits:** **Grade (written part):**

Problem 1 (18 points) Consider a random variable x with pdf

$$f_{\mathsf{x}}(x) = \begin{cases} 2(1-x), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases} .$$

a) Sketch the pdf of x.

b) Calculate the mean and variance of x.

c) Let $\mathsf{y} = \mathsf{x}_1 - \mathsf{x}_2$, where x_1 and x_2 are iid random variables with pdf $f_{\mathsf{x}}(x)$. Find $f_{\mathsf{y}}(y)$.

d) Calculate the mean and variance of y without using its pdf $f_{\mathsf{y}}(y)$.

e) Find the probability $\mathrm{P}\{|\mathsf{y}| > 1/2\}$.

Problem 2 (18 points) Consider two random variables x, y with joint pdf

$$f_{\mathsf{x},\mathsf{y}}(x, y) = \begin{cases} C, & (x-1)^2 + (y-1)^2 \le 1 \\ 0, & \text{otherwise} \end{cases} ,$$

where C is a constant. A sketch of the joint pdf of x, y is shown in Figure 1.

a) Calculate C.

b) Is the random variable x Rayleigh distributed? Justify your answer without calculating $f_{\mathsf{x}}(x)$ explicitly.

c) Find and sketch the conditional pdf $f_{\mathsf{x}|\mathcal{A}}(\mathsf{x}|\mathcal{A})$ for the event $\mathcal{A} = \{\mathsf{y} = 1\}$.

d) Find the conditional expectation $\mathrm{E}\{\mathsf{x}|\mathsf{y} = y\}$ for $y \in [0, 2]$.

e) Find the conditional probability $\mathrm{P}\{\mathsf{y} < \mathsf{x}|\mathsf{x} \ge 1\}$.

f) Find the pdf of the random variable $\mathsf{z} = \sqrt{(\mathsf{x}-1)^2 + (\mathsf{y}-1)^2}$.

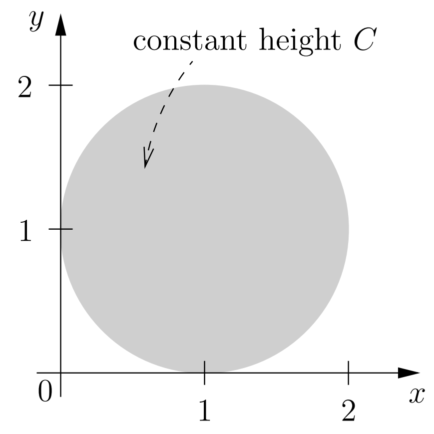

Figure 1: Sketch of the joint pdf $f_{\mathsf{x},\mathsf{y}}(x, y)$.

Problem 3 (19 points) Consider the random process

$$\mathsf{w}[n] = \mathsf{x}[n]\cos(\theta_0 n) + \mathsf{y}[n]\sin(\theta_0 n) + \mathsf{v}[n]\,,$$

where $\theta_0 > 0$ is a constant. The random processes $\mathsf{x}[n]$, $\mathsf{y}[n]$, and $\mathsf{v}[n]$ are jointly WSS.

a) Is the random process $\mathsf{w}[n]$ wide-sense cyclostationary? If yes, specify the period N_0.

b) Find conditions on $\mathsf{x}[n]$, $\mathsf{y}[n]$, and $\mathsf{v}[n]$ such that $\mathsf{w}[n]$ is a WSS random process.

c) With the conditions of the previous subtask applied to $\mathsf{w}[n]$, find its power spectral density (PSD) in terms of the (cross) PSDs of $\mathsf{x}[n]$, $\mathsf{y}[n]$, and $\mathsf{v}[n]$.

d) Specialize the result of the previous subtask for the case that $\mathsf{x}[n]$ and $\mathsf{y}[n]$ are also uncorrelated.

Problem 4 (15 points) Let $\mathbf{x} = (\mathsf{x}_1\ \mathsf{x}_2\ \mathsf{x}_3)^T$ be a zero-mean random vector with correlation matrix

$$\boldsymbol{R}_{\mathbf{x}} = \begin{pmatrix} P_{\mathsf{x}_1} & \alpha & 0 \\ \alpha & P_{\mathsf{x}_2} & \beta \\ 0 & \beta & P_{\mathsf{x}_3} \end{pmatrix}.$$

Hint for c) – e): Recall the spectral decomposition of a symmetric $m \times m$ matrix $\boldsymbol{A}$ with distinct eigenvalues λ_k and orthonormal eigenvectors $\boldsymbol{u}_k$ $(k = 1, 2, \ldots, m)$:

$$\boldsymbol{A} = \sum_{k=1}^{m} \lambda_k \boldsymbol{P}_k, \quad \text{with} \quad \boldsymbol{P}_k = \boldsymbol{u}_k \boldsymbol{u}_k^T,$$

where $\boldsymbol{P}_k$ is a projection matrix, projecting onto the subspace spanned by $\boldsymbol{u}_k$. A projection matrix $\boldsymbol{P}$ is symmetric and idempotent, i.e., $\boldsymbol{P}^2 = \boldsymbol{P}$. For $P_{\mathsf{x}} \triangleq P_{\mathsf{x}_1} = P_{\mathsf{x}_2} = P_{\mathsf{x}_3} = 1$ and $\alpha = \beta = 1/4$, $\boldsymbol{R}_{\mathbf{x}}$ has the following spectral decomposition:

$$\boldsymbol{P}_1 = -\frac{1}{4}\begin{pmatrix} -1 & \sqrt{2} & -1 \\ \sqrt{2} & -2 & \sqrt{2} \\ -1 & \sqrt{2} & -1 \end{pmatrix}, \quad \boldsymbol{P}_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{P}_3 = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix},$$

with the corresponding eigenvalues $\lambda_1 = 1 - \sqrt{2}/4$, $\lambda_2 = 1$, $\lambda_3 = 1 + \sqrt{2}/4$.

a) Specify the possible range of values for α and β.

b) Specify the largest possible value of β and the corresponding value of α.

c) Assume $P_{\mathsf{x}} = 1$, $\alpha = \beta = 1/4$ for the remainder of this problem and let $\mathbf{y} = \boldsymbol{F}\mathbf{x}$. Find the filtering matrix $\boldsymbol{F}$ such that the subspace corresponding to λ_1 is suppressed and all other subspace components remain unchanged, i.e., $\boldsymbol{P}_1\mathbf{y} = \mathbf{0}$, $\boldsymbol{P}_2\mathbf{y} = \boldsymbol{P}_2\mathbf{x}$, $\boldsymbol{P}_3\mathbf{y} = \boldsymbol{P}_3\mathbf{x}$. *Hint: do not calculate the matrix products explicitly; use the properties of projection matrices instead.*

d) Calculate the correlation matrix $\boldsymbol{R}_{\mathbf{y}}$.

e) Is the filtering using $\boldsymbol{F}$ equivalent to the truncated Karhunen-Loève transformation (KLT) of $\mathbf{x}$, where the eigenvector corresponding to the smallest eigenvalue of $\boldsymbol{C}_{\mathbf{x}}$ is not used in the re-synthesis? Justify your answer without explicitly calculating the KLT of $\mathbf{x}$.